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Original article

Efficient entropy estimation for inverted exponentiated Pareto distribution using ranked set sampling: A comparative study

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ABSTRACT

Entropy, a key concept in information theory, measures the degree of unpredictability or uncertainty present in a random variable or system. It plays a vital role across various disciplines, including communication theory, thermodynamics, and statistical mechanics. On the other hand, Ranked Set Sampling (RSS) provides an effective approach to mitigating the challenges associated with costly or complex measurement procedures. Given the wide-ranging applications of the inverted exponentiated Pareto distribution, this study investigates the estimation of its parameters and various entropy measures, encompassing Havrda and Charvát, Tsallis, Rényi, and Arimoto. We examine the performance of these estimators under both RSS and simple random sampling (SRS) frameworks.To tackle this task, seven classical estimation techniques are employed: maximum product spacing, least squares, Kolmogorov, Anderson-Darling, weighted least squares, maximum likelihood, and Cramér-von Mises. Using an equal number of measured units, simulation studies evaluates the performance of estimators derived from SRS and RSS, considering both perfect and imperfect ranking scenarios. Three evaluation criteria are adopted for comparison: relative efficiency, mean squared error, and absolute bias. In assessing the estimated quality of RSS and SRS, the Kolmogorov technique appears beneficial in most cases, based on numerical results. In terms of estimation accuracy, RSS consistently performs better than SRS, regardless of whether the ranking is perfect or imperfect. Additionally, compared to imperfect ranking method, perfect ranking produces estimates that are more accurate. The advantage of the RSS design over the SRS design is further supported by real data results that indicate the tensile strength measures in GPA carbon fibers.

1. Introduction

1.1. Entropy measures

Entropy is a key concept in measuring the level of uncertainty associated with a random variable. Although thermodynamics was the original background for its development, its significance extends far beyond its original scope, as it finds applications in a wide range of disciplines, including biology, statistics, physics, chemistry, economics, insurance, and finance. Although the phrase "information theory" lacks a universally accepted definition, it can be informally defined as the study of systems governed by probabilistic rules. Entropy, in this context, measures the amount of information present in a sample. Probability distribution analysis is a fundamental component of statistics. A certain amount of uncertainty is inherent in all probability distributions, and entropy a quantitative indicator of this uncertainty. It also measures the degree of disorder or randomness in a system. Entropy is maximized in systems with several equally likely random states, and it reaches zero when the system is in a state of complete certainty and clarity. Shannon [1] introduced the concept of entropy to the statistical community, adapting it from physics. It serves as a gauge of the ambiguity surrounding an event and is widely known as Shannon entropy.

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This study focuses on four entropy measures: Rényi [2], Havrad and Charvát [3], Tsallis [4], and Arimoto [5].

The issue of estimating entropy functions for various distributions has been addressed by many researchers. Kang et al. [6] investigated entropy estimators for the double exponential distribution using multiple Type II censored samples. Cho et al. [7] proposed entropy estimators for the Rayleigh distribution under doubly- generalized Type II hybrid censoring. Using the hazard rate function, Baratpour et al. [8] studied the entropy of upper record values and provided several upper and lower bounds for this entropy. Liu and Gui [9] explored Shannon entropy for the Lomax distribution under a generalized progressively hybrid censoring scheme. Chacko and Asha [10] derived maximm likelihood (ML) estimators for Rényi entropy and q-entropy of the Rayleigh distribution based on record values. Hassan and Zaky [11] examined point and interval estimation of Shannon entropy for the inverse Weibull distribution under multiple censoring scenarios. Entropy estimation problems for Burr XII, log-logistic, inverse Weibull, and generalized inverse exponential distributions under progressive censoring schemes were investigated by Wang and Gui [12], Shrahili et al. [13], Abdelall and Abdullah [14], and Gong and Yin [15], respectively. Helmy et al. [16–18] studied ML and Bayesian estimators for entropy under unified hybrid censoring and generalized Type-I hybrid censored schemes. Shi et al. [19] estimated entropy for the generalized Bilal distribution using adaptive Type- II progressively hybrid censored samples. Hassan and Zaki [20] proposed a Bayesian estimator for Shannon entropy and the corresponding credible interval using symmetric and asymmetric loss functions with record values. Hassan et al. [21–24] investigated entropy estimation for the power-function and Pareto distributions in the presence of outliers. Hassan et al. [25] also analyzed ML and Bayesian estimators for several entropy measures based on generalized Type II hybrid censored data.

1.2. Ranked set sampling method

Simple random sampling (SRS) is a popular method for developing statistical techniques. However, small sample sizes may yield unrepresentative samples. Alternative techniques, such as cluster, stratified, and systematic sampling, have been developed to overcome this. To enhance validity, these techniques make use of past understanding of the population's composition. Ranked set sampling (RSS) was introduced in [26] to minimize the number of measurements while preserving accuracy. In investigations when it is difficult to achieve accurate measurements because of significant expenditures (financial, time, labor, and organizational), RSS is very helpful. Even if it could be challenging to measure exactly, the variable of interest is frequently reasonably easy to rank (order) at little or no additional cost. Visual examination, preliminary data, historical sample data, or other low-cost techniques that do not call for precise measurements can all be used to determine this rating. RSS has been effectively used in environmental [27], quality control [28], fisheries research [29], engineering applications [30,31], reliability studies [32], and medical research [33]. Takahasi and Wakimoto [34] provided the mathematical theory for RSS. Even in the presence of ranking error, Dell and Clutter [35] showed that RSS outperforms SRS in efficiency. Extreme RSS, a variation of RSS, was introduced by Samawi et al. [36] in order to decrease ranking error. Muttlak [37] looked into a different RSS technique called the median RSS. Quantile estimation from a population with a known mean was examined by Mahdizadeh and Arghami [38] using an RSS.

Several studies have explored uncertainty measures based on RSS and its extensions. For instance, Jozani and Ahmadi [39] investigated the concepts of uncertainty and information content in RSS data under both perfect and imperfect ranking scenarios. The uncertainty structure and Fisher information content of partially rank-ordered set sampling designs were examined by Hatefi and Jozani [40]. For RSS and multistage RSS designs, Mahdizadeh and Arghami [41] suggested uniformity tests based on sample entropy. Mahdizadeh [42] discussed testing exponentiality based on sample entropy under some RSS-based designs. Zamanzade and Mahdizadeh [43] have proposed a number of nonparametric entropy estimators within the RSS framework. Mahdizadeh [44] investigated the Rayleigh distribution's goodness-of-fit test using the Kullback-Leibler information criterion. Under the advanced RSS scheme, Al-Omari et al. [45] developed an entropy estimator for the generalized inverse exponential distribution.

The following steps explain the RSS design:

i. From the underlying population, select t SRS of size t (set size).

$$\begin{matrix} X_{11}, & X_{12}, & \ldots, & X_{1t} \\ X_{21}, & X_{22}, & \ldots, & X_{2t} \\ \vdots & \vdots & \cdots, & \vdots \\ X_{t1}, & X_{t2}, & \cdots, & X_{tt} \end{matrix}$$

ii. Visually arrange the units in each set of size (t) from smallest to largest, or use any free technique as:

$$\begin{matrix} X_{1(1)}, & X_{1(2)}, & \ldots, & X_{1(t)} \\ X_{2(1)}, & X_{2(2)}, & \ldots, & X_{2(t)} \\ \vdots & \vdots & \cdots, & \vdots \\ X_{t(1)}, & X_{t(2)}, & \cdots, & X_{t(t)} \end{matrix}$$

iii. From the h_1^{th}, set ($h_1 = 1, 2,\ldots, t$), choose the h_1^{th} order statistic (in brackets), where the h_1^{th}, data point (measured unit) acquired in the h_2^{th} cycle is denoted by $Y_{h_2h_1} = X_{h_2(h_1)}$, $h_1 = 1, 2,\ldots, t$, and $h_2 = 1, 2,\ldots, d$

$$\begin{matrix} [Y_{11}], & Y_{12}, & \ldots, & Y_{1t} \\ Y_{21}, & [Y_{22}], & \ldots, & Y_{2t} \\ \vdots & \vdots & \cdots, & \vdots \\ Y_{t1}, & Y_{t2}, & \cdots, & [Y_{tt}] \end{matrix}$$

iv. An RSS of size $t^* = d \times t$ can be obtained by repeating Steps (i)–(iii) d times (cycles).

The chosen RSS units are, represented by $(Y_{11}, Y_{22}, \ldots, Y_{tt})$, where $Y_{h_1h_2}$, is the largest unit in a collection of size (t) in the (d) cycle. It should be noted that even though we chose t^2 units, we only measured t of them; despite their differing distributions, these units are independent since they are chosen from distinct sets. Let $Y_{h_2h_1}$ represents the order statistics of the h_1^{th} sample, with $h_1 = 1, 2, \ldots, t$ in h_2^{th} cycle, where $h_2 = 1,2,\ldots, d$. Assuming perfect ranking, the probability density function (PDF) of $Y_{h_2h_1}$ is given by:

$$f_{Y_{h_1h_2}}\left(y_{h_1h_2}\right) = \frac{t!}{(h_1-1)!(t-h_1)!} f\left(y_{h_1h_2}\right)\left[F\left(y_{h_1h_2}\right)\right]^{h_1-1}\left[1-F\left(y_{h_1h_2}\right)\right]^{t-h_1}; \quad -\infty < y_{h_1h_2} < \infty. \tag{1}$$

Wolfe [46] argued that set sizes (t) larger than five are likely to lead to a significant rise in ranking errors, which could reduce RSS's performance advantages. Although perfect rankings are the ideal goal of any RSS methodology, they are sometimes impossible to achieve in real-world scenarios. Therefore, it is crucial to assess the impact of imperfect rankings on statistical processes. The most thorough method for tackling this issue is creating statistical models that specifically measure and account for the uncertainty present in the ranking procedure. For more information about imperfect ranking, the reader can refer to Frey [47,48].

1.3. Inverted exponetaited Pareto distribution

Inverse distributions are crucial for statistical modeling and analysis,

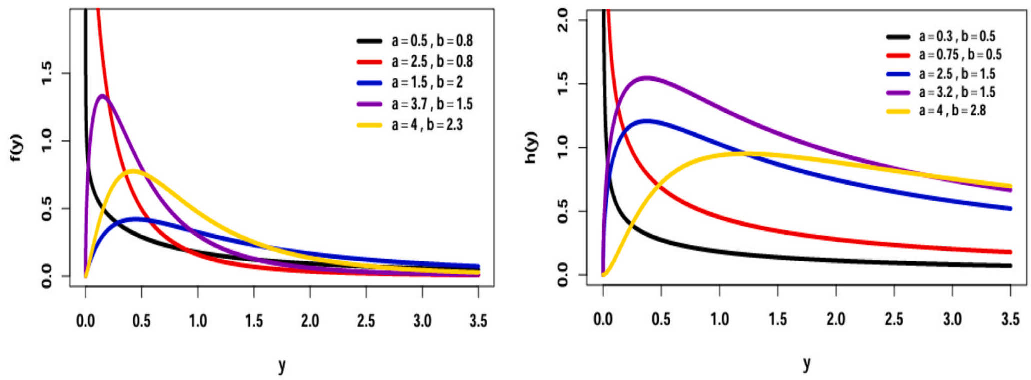

Fig. 1. Plots of the PDF and HRF for the IEP distribution.

particularly when dealing with skewed or heavy-tailed data. These distributions are created by inverting the order of the original distribution, which results in a new distribution with distinct properties. This transformation is often necessary to capture the unique characteristics of specific datasets, such as those found in sectors like reliability engineering, insurance, and finance. An inverse exponentiated class of probability distributions was recently introduced and studied by Ghitany et al. [49]. The PDF of this class is defined by:

$$f(y) = \frac{ab}{y^2} W'(y^{-1}) \ \left(1 - e^{-bW(y^{-1})}\right)^{a-1} e^{-bW(y^{-1})}; \quad y, a, b > 0, \tag{2}$$

where $W(.)$ is an increasing function provided that $W(0) = 0$ and $W(\infty) = \infty$, while a and b are model parameters. This class includes a number of popular models, including the inverted exponentiated Pareto (IEP) distribution, which is the focus of this study. The PDF of the IEP distribution is produced by setting $W(y^{-1}) = \log(1 + y^{-1})$ in Eq. (2) as follows:

$$f(y) = aby^{b-1} \ (1 + y)^{-(b+1)} \left[1 - \left(1 + y^{-1}\right)^{-b}\right]^{a-1}; \quad y, a, b > 0, \tag{3}$$

where a and b are the shape parameters. The cumulative distribution function (CDF) of the IEP distribution is given by:

$$F(y) = 1 - \left[1 - \left(1 + y^{-1}\right)^{-b}\right]^{a}; \quad y, a, b > 0, \tag{4}$$

The hazard rate function (HRF) is of the form

$$h(y) = aby^{b-1} \ (1 + y)^{-(b+1)} \left[1 - \left(1 + y^{-1}\right)^{-b}\right]^{-1}.$$

Fig. 1 displays the PDF and HRF plots of various parameter combinations for IEP distribution. The PDF of the IEP distribution is right-skewed, unimodal, increasing, reversed J-shaped, and decreasing. Additionally, the HRF of the IEP distribution could be increasing, decreasing, reversed J-shaped or upside-down. It is also seen that the behavior can be non-monotone in nature. Applications of the IEP distribution could be found in a variety of fields, including fatigue failure analysis, degradation tests, the investigation of mechanical or electrical component reliability, and mortality data examination. A useful tool for reliability and lifetime studies, this distribution is similar to other widely used models such as the generalized exponential distribution, exponentiated moment exponential distribution, and inverted exponential Rayleigh distribution, as mentioned by Pradhan and Kundu [50]. An investigation of the estimate of unknown parameters under progressive Type-II censoring was conducted by Maurya et al. [51]. Using record values, Shaikh and Patel [52] investigated parameter estimation. Estimation of multicomponent stress-strength reliability based on censoring samples was discussed by [53,54].

1.4. Motivation of study

To the best of our knowledge, there isn't any research that focuses on estimating uncertainty measures using RSS methodologies. This study investigates the estimation of different entropy measures for the IEP distribution based on RSS. The IEP distribution has important uses in a variety of domains, such as mortality data analysis, fatigue failure analysis, degradation testing, and reliability evaluation of mechanical and electrical components. Measures of uncertainty are essential because they help make well-informed decisions by quantifying the degree of confidence in estimation. The motivations for this work are as follows:

- Obtain the parameter estimators of the IEP distribution as well as the estimators of some entropy measures. The recommended measures are Havrda and Charvát, Tsallis, Rényi and Arimoto measures.
- Using seven different estimation methods, such as Kolmogorov (KM), Anderson-Darling (AD), least squares (LS), maximum product spacing (MPS), Cramér-von Mises (CVM), ML, and weighted LS (WLS), the parameter and entropy estimators are obtained using RSS and SRS techniques.
- Monte Carlo simulations are used to assess the performance of various entropy metrics under SRS and RSS (perfect and imperfect ranking) utilizing some accuracy measures including relative efficiencies (REs), mean squared errors (MSEs), and absolute biases (ABs).
- The inferential approaches presented in this paper were applied to real-world data to demonstrate their effectiveness.

The structure of this document is as follows: In Section 2, mathematical expressions for several entropy metrics are presented. The IEP distribution's parameter and entropy estimators based on ML and MPS methods are covered in Sections 3 and 4, respectively, for both RSS and SRS.

In Section 5, the estimators of parameters and entropy measures based on LS and WLS methods are examined. Section 6 examines the AD and CVM estimators of entropy measurements and parameters.

Section 7 describes the outcomes of numerical simulations used to assess the suggested estimators' performance. Section 8 uses RSS and SRS to analyze KM estimators of parameters and entropy measurements. Section 9 uses a real-world dataset to illustrate the method's applicability.

The paper is concluded in Section 10 with a summary of the main conclusions and suggestions for future studies.

2. Formula for entropy metrics

This section provides the analytical expressions for the Rényi entropy, denoted by γ_1, Havrda-Charvát entropy, denoted by γ_2, Tsallis entropy, denoted by γ_3, and Arimoto entropy, denoted by denoted by γ_4, for the IEP distribution.

The Rényi entropy of order υ, where $\upsilon > 0$, and $\upsilon \neq 1$, is defined as follows:

$$\gamma_1 = \frac{1}{1-\upsilon} \log\left[\int_{-\infty}^{\infty} f(y)^{\upsilon} dy\right]. \tag{5}$$

The Rényi entropy of the IEP distribution is obtained by inserting PDF (3) in Eq. (5) as follows:

$$\gamma_1 = \frac{1}{1-\upsilon} \log\left[\int_{0}^{\infty} (ab)^{\upsilon} y^{\upsilon(b-1)} \ (1+y)^{-\upsilon(b+1)}\left[1-\left(1+y^{-1}\right)^{-b}\right]^{(a-1)\upsilon} dy\right]. \tag{6}$$

Using binomial expansion in the integrated part of Eq. (6) leads to

$$(f(y))^{\upsilon} = \sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) \int_0^{\infty} y^{\upsilon(b-1)+bl} \ (1+y)^{-\upsilon(b+1)-bl} dy, \tag{7}$$

where $\eta_l(\tau,\upsilon) = (-1)^l (ab)^{\upsilon} \binom{(a-1)\upsilon}{l}$ and $\tau = (a,b)$. The integral part in Eq. (7) is the beta function of the second kind that gives the following formula:

$$(f(y))^{\upsilon} = \sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) B(\upsilon(b-1)+bl+1, 2\upsilon-1), \tag{8}$$

where B(.,.) is the beta function. Then by substituting Eq. (8) in Eq. (6), the Rényi entropy of the IEP distribution is given by

$$\gamma_1 = \frac{1}{1-\upsilon} \log\left[\sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) B(\upsilon(b-1)+bl+1, 2\upsilon-1)\right]. \tag{9}$$

The Tsallis entropy of order υ, $\upsilon > 0$, and $\upsilon \neq 1$, is defined by:

$$\gamma_2 = \frac{1}{\upsilon-1}\left[1-\int_{-\infty}^{\infty} f(y)^{\upsilon} dy\right]. \tag{10}$$

Substituting Eq. (8) in Eq. (10), then the Tsallis entropy of the IEP distribution is given by:

$$\gamma_2 = \frac{1}{\upsilon-1}\left[1-\sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) B(\upsilon(b-1)+bl+1, 2\upsilon-1)\right]. \tag{11}$$

The Havrda and Charvát Entropy of order υ, where $\upsilon > 0$, and $\upsilon \neq 1$, is defined by:

$$\gamma_3 = \frac{1}{2^{1-\upsilon}-1}\left[\int_{-\infty}^{\infty} f(y)^{\upsilon} dy - 1\right]. \tag{12}$$

Substituting Eq. (8) in Eq. (12), then the Havrda and Charvát entropy of the IEP distribution is given by:

$$\gamma_3 = \frac{1}{2^{1-\upsilon}-1}\left[\sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) B(\upsilon(b-1)+bl+1, 2\upsilon-1) - 1\right]. \tag{13}$$

The Arimoto entropy of order υ, $\upsilon > 0$, and $\upsilon \neq 1$, is defined by:

$$\gamma_4 = \frac{\upsilon}{1-\upsilon}\left[\left(\int_{-\infty}^{\infty} f(y)^{\upsilon} dy\right)^{1/\upsilon} - 1\right]. \tag{14}$$

Substituting Eq. (8) in Eq. (14), then the Arimoto entropy of the IEP distribution is given by:

$$\gamma_4 = \frac{\upsilon}{1-\upsilon}\left[\left(\sum_{l=0}^{\infty} \eta_l(\tau,\upsilon) B(\upsilon(b-1)+bl+1, 2\upsilon-1)\right)^{1/\upsilon} - 1\right]. \tag{15}$$

3. Maximum likelihood estimator

In this section, we focus on the ML estimators of the parameters a and b of the IEP distribution. Subsequently, we utilize the invariance property of ML estimators to derive the corresponding estimators for the entropy measures γ_1, γ_2, γ_3, and γ_4. These estimators are obtained under both RSS and SRS frameworks.

3.1. Estimators based on RSS

The ML estimators $\widehat{a}^{\text{ML}}$ of a and $\widehat{b}^{\text{ML}}$ of b of the IEP distribution are obtained based on RSS. Furthermore the ML estimators of the entropy measures $\widehat{\gamma}_1^{\text{ML}}$of γ_1, $\widehat{\gamma}_2^{\text{ML}}$of γ_2, $\widehat{\gamma}_3^{\text{ML}}$of γ_3, and $\widehat{\gamma}_4^{\text{ML}}$of γ_4, from the IEP distribution are obtained. To get these estimators let $Y_{h_1h_2} = \{Y_{h_1h_2}, h_1 = 1, 2, \ldots, t, h_2 = 1, 2, \ldots, d\}$ be the RSS of size $t^* = t \times d$ where d is cycles count and t is the set size. The PDF of $Y_{h_1h_2}$ of the h_1-th order statistics, assuming the ordering of the observations is perfect, is provided by substituting the PDF (3) and CDF (4) in Eq. (1) as:

$$f_{Y_{h_1h_2}}\left(y_{h_1h_2}\right) = \frac{t!\,ab y_{h_1h_2}^{b-1}\left(1+y_{h_1h_2}\right)^{-(b+1)}}{(h_1-1)!(t-h_1)!}\left[M\left(y_{h_1h_2},b\right)\right]^{a(t-h_1+1)-1}\left[1 - \left[M\left(y_{h_1h_2},b\right)\right]^a\right]^{h_1-1}; 0 < y_{h_1h_2} < \infty, \tag{16}$$

where $M(y_{h_1h_2},b) = 1-\left(1+y_{h_1h_2}^{-1}\right)^{-b}$. The likelihood function (LF) for an RSS under perfect ranking assumption, is obtained from PDF (16) as follows:

$$l(\tau) \propto \prod_{h_2=1}^{t}\prod_{h_1=1}^{d} ab y_{h_1h_2}^{b-1}\left(1+y_{h_1h_2}\right)^{-(b+1)}\left[M\left(y_{h_1h_2},b\right)\right]^{a(t-h_1+1)-1}\left[1 - \left[M\left(y_{h_1h_2},b\right)\right]^a\right]^{h_1-1}, \tag{17}$$

where $\tau \equiv (a,b)$.The log-LF is then obtained by:

$$l_1^*(\tau) \propto t^* \log\left(ab\right) - \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\left(b+1\right)\log\left(1+y_{h_1h_2}\right) + \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}[a(t-h_1+1)-1]\log\left[M\left(y_{h_1h_2},b\right)\right] + \sum_{h_2=1}^{t} \times \sum_{h_1=1}^{d}(b-1)\log\left(y_{h_1h_2}\right) + \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\left(h_1-1\right)\log\left[1-\left[M\left(y_{h_1h_2},b\right)\right]^a\right].$$

Differentiating the log-LF with respect to the parameters a and b

$$\frac{\partial l_1^*(\tau)}{\partial a} = \frac{t^*}{a} + \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\left(t-h_1+1\right)\log\left[M\left(y_{h_1h_2},b\right)\right] - \sum_{h_2=1}^{t} \times \sum_{h_1=1}^{d}\frac{\left(h_1-1\right)\log\left[M\left(y_{h_1h_2},b\right)\right]}{\left[M\left(y_{h_1h_2},b\right)\right]^{-a}-1}, \tag{18}$$

and,

$$\frac{\partial l_1^*(\tau)}{\partial b} = \frac{t^*}{b} - \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\log\left(1+y_{h_1h_2}\right)$$

$$+\sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\frac{[a(t-h_1+1)-1]M\prime\left(y_{h_1h_2},b\right)}{M\left(y_{h_1h_2},b\right)} + \sum_{h_2=1}^{t}\sum_{h_1=1}^{d}\log\left(y_{h_1h_2}\right) - \sum_{h_2=1}^{t}$$
$$\times\sum_{h_1=1}^{d}\frac{\left(h_1-1\right)a\left[M\left(y_{h_1h_2},b\right)\right]^{a-1}M\prime\left(y_{h_1h_2},b\right)}{\left[1-\left[M\left(y_{h_1h_2},b\right)\right]^{a}\right]}, \tag{19}$$

where $M\prime(y_{h_1h_2},b) = -\left(1+y_{h_1h_2}^{-1}\right)^{-b}\log\left(1+y_{h_1h_2}^{-1}\right)$.

Due to the complexity of Eqs. (18) and (19), a closed-form solution for the ML estimators is generally not available. Therefore, numerical optimization techniques are necessary to find $\widehat{a}^{ML}$ of a and $\widehat{b}^{ML}$ of b that maximize the LF. Further, the ML estimators of entropy measures $\widehat{\gamma}_1^{ML}$ of γ_1, $\widehat{\gamma}_2^{ML}$ of γ_2, $\widehat{\gamma}_3^{ML}$ of γ_3, and $\widehat{\gamma}_4^{ML}$ of γ_4, are obtained after inserting $\widehat{a}^{ML}$ and $\widehat{b}^{ML}$ in Eqs. (9), (11), (13), and (15), respectively, utilizing the invariance property of ML estimators.

3.2. Estimators based on SRS

This sub-section gives the ML estimators of parameters a and b, denoted respectively by $\widetilde{a}^{ML}$ and $\widetilde{b}^{ML}$ of the IEP distribution using the SRS method. Furthermore the ML estimators of the entropy measures $\widetilde{\gamma}_1^{ML}$, of γ_1, $\widetilde{\gamma}_2^{ML}$, of γ_2, $\widetilde{\gamma}_3^{ML}$, of γ_3, and $\widetilde{\gamma}_4^{ML}$, of γ_4, from the IEP distribution are obtained. To get these estimators suppose that $y_1, y_2, \ldots, y_{t^*}$ be the SRS of size t^* drawn from the IEP distribution with PDF (3). The log-LF of the IEP distribution under SRS is as below:

$$l_2^*(\tau) = t^*\log\left(ab\right) + \sum_{j=1}^{t^*}(b-1)\log\left(y_j\right) - \sum_{j=1}^{t^*}(b+1)\log\left(1+y_j\right)$$
$$+\sum_{j=1}^{t^*}(a-1)\log\left[1-\left(1+y_j^{-1}\right)^{-b}\right]. \tag{20}$$

The system of equations that follows is obtained by differentiating the log-LF (20) with respect to the parameters a and b and setting these derivatives to zero:

$$\frac{\partial l_2^*(\tau)}{\partial a} = \frac{t^*}{a} + \sum_{j=1}^{t^*}\log\left[1-\left(1+y_j^{-1}\right)^{-b}\right] = 0, \tag{21}$$

$$\frac{\partial l_2^*(\tau)}{\partial b} = \frac{t^*}{b} + \sum_{j=1}^{t^*}\left[\log\left(y_j\right) - \log\left(1+y_j\right)\right]$$
$$+\sum_{j=1}^{t^*}\frac{\left(a-1\right)\left(1+y_j^{-1}\right)^{-b}\log\left(1+y_j^{-1}\right)}{\left[1-\left(1+y_j^{-1}\right)^{-b}\right]} = 0. \tag{22}$$

The ML estimator $\widetilde{a}^{ML}$ of a as a function of b, say $\widetilde{a}^{ML}(b)$, is produced by using Eq. (21) as follows:

$$\widetilde{a}^{ML}(b) = \frac{-t^*}{\sum_{j=1}^{t^*}\log\left[1-\left(1+y_j^{-1}\right)^{-b}\right]} = \widetilde{D}^{ML}(b). \tag{23}$$

Substituting Eq. (23) into Eq. (22), we obtain

$$\frac{t^*}{b} + \sum_{j=1}^{t^*}\left[\log\left(y_j\right) - \log\left(1+y_j\right)\right]$$
$$+\sum_{j=1}^{t^*}\frac{\left(\widetilde{D}^{ML}(b)-1\right)\left(1+y_j^{-1}\right)^{-b}\log\left(1+y_j^{-1}\right)}{\left[1-\left(1+y_j^{-1}\right)^{-b}\right]} = 0. \tag{24}$$

By employing an iterative technique to solve Eq. (24), we obtain $\widetilde{b}^{ML}$. After that, by substituting $\widetilde{b}^{ML}$ in Eq. (23), we obtain $\widetilde{a}^{ML}$. Further, the ML estimators of entropy measures $\widetilde{\gamma}_1^{ML}$ of γ_1, $\widetilde{\gamma}_2^{ML}$ of γ_2, $\widetilde{\gamma}_3^{ML}$ of γ_3, and $\widetilde{\gamma}_4^{ML}$ of γ_4 are obtained after inserting $\widetilde{a}^{ML}$ and $\widetilde{b}^{ML}$ in Eqs. (9), (11), (13), and (15), respectively, utilizing the invariance property of the ML estimators.

4. Maximum product spacing

Instead of ML approache, Cheng and Amin [55] were the first to utilize the MPS technique for parameter estimation. Notably, Ranneby [56] had earlier recognized MPS as a viable approximation for the Kullback-Leibler information measure, a fundamental concept in information theory. This section investigates the derivation of MPS estimators $\widehat{a}^{MPS}$ of a and $\widehat{b}^{MPS}$ of b of the IEP distribution. We utilize both RSS and SRS methods for this aim. Additionally, the MPS estimators are used to obtain the relevant estimators for the entropy measures γ_1, γ_2, γ_3, and γ_4.

4.1. Estimators based on RSS

The MPS estimators $\widehat{\gamma}_1^{MPS}$ of γ_1, $\widehat{\gamma}_2^{MPS}$ of γ_2, $\widehat{\gamma}_3^{MPS}$ of γ_3, and $\widehat{\gamma}_4^{MPS}$ of γ_4, $\widehat{a}^{MPS}$ of a and $\widehat{b}^{MPS}$ of b of the IEP distribution are obtained based on RSS.

Suppose that $Y_{(1:t^*)}, \ldots, Y_{(t^*:t^*)}$ be the RSS of size $t^* = t\times d$ where d is cycles count and t is the set size. The spacing of a random sample of size $t^* = t\times d$ is given as $\Lambda_k(\tau) = H(y_{(k:t^*)}) - H(y_{(k-1:t^*)}), k = 1,2,\ldots,t^*$, $H(y_{(0)}) = 0$, $H(y_{(t^*+1:\ t^*)}) = 1$. The proposed estimator $\widehat{a}^{MPS}$ of a and $\widehat{b}^{MPS}$ of b is computed by maximizing the following equation

$$M^* = \frac{1}{(1+t^*)}\sum_{k=1}^{t^*+1}\ln(\Lambda_k(\tau)). \tag{25}$$

Or the MPS estimators $\widehat{a}^{MPS}$ and $\widehat{b}^{MPS}$ are produced by solving the following equations:

$$\frac{\partial M^*}{\partial a} = \frac{1}{t^*+1}\sum_{k=1}^{t^*+1}\frac{1}{\Lambda_k(\tau)}\left\{1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^{a}\right.$$
$$\left.-\left\{1-\left[1-\left(1+y_{(k-1:t^*)}^{-1}\right)^{-b}\right]^{a}\right\}\right\}\pi_1\left(y_{(k:t^*)}|\tau\right) - \pi_1\left(y_{(k-1:t^*)}|\tau\right) = 0,$$

$$\frac{\partial M^*}{\partial b} = \frac{1}{t^*+1}\sum_{k=1}^{t^*+1}\frac{1}{\Lambda_k(\tau)}\left\{1-\left[1-\left(1+y_{k:t^*}^{-1}\right)^{-b}\right]^{a}\right.$$
$$\left.-\left\{1-\left[1-\left(1+y_{k-1:t^*}^{-1}\right)^{-b}\right]^{a}\right\}\right\}\pi_2\left(y_{(k:t^*)}|\tau\right) - \pi_2\left(y_{(k-1:t^*)}|\tau\right) = 0,$$

where

$$\pi_1\left(y_{(k:t^*)}|\tau\right) = \frac{\partial}{\partial a}[H(y_{(k:t^*)}|\tau)]$$
$$= -\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^{a}\ln\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right], \tag{26}$$

$$\pi_2\left(y_{(k:t^*)}|\tau\right) = \frac{\partial}{\partial b}[H(y_{(k:t^*)}|\tau)] = -a\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^{a-1}\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\ln\left(1+y_{(k:t^*)}^{-1}\right), \tag{27}$$

where $\pi_1(y_{(k-1:t^*)}|\tau)$, and $\pi_2(y_{(k-1:t^*)}|\tau)$, share the same expression in Eqs. (26) and (27) by replacing $y_{(k:t^\bullet)}$ with $y_{(k-1:t^\bullet)}$. Further, the MPS estimators $\widehat{\gamma}_1^{MPS}$, $\widehat{\gamma}_2^{MPS}$, $\widehat{\gamma}_3^{MPS}$, and $\widehat{\gamma}_4^{MPS}$ are obtained after inserting $\widehat{a}^{MPS}$ and $\widehat{b}^{MPS}$ in Eqs. (9), (11), (13), and (15), respectively.

4.2. Estimators based on SRS

The MPS estimators $\widetilde{\gamma}_1^{MPS}$ of γ_1, $\widetilde{\gamma}_2^{MPS}$ of γ_2, $\widetilde{\gamma}_3^{MPS}$ of γ_3, and $\widetilde{\gamma}_4^{MPS}$ of γ_4, $\widetilde{a}^{MPS}$ of a and $\widetilde{b}^{MPS}$ of b of the IEP distribution are obtained based on SRS. To obtain these estimators, let $y_{(1)}, \ldots, y_{(t^*)}$ be the SRS of size t^*. The spacing of a random sample of size t^* is given as $\Lambda_i(\tau) = H(y_{(i)}) - H(y_{(i-1)}), i = 1, 2, \ldots, t^*, H(y_{(0)}) = 0, \quad H(y_{(t^*+1)}) = 1$. The estimators $\widetilde{a}^{MPS}$ of a and $\widetilde{b}^{MPS}$ of b is computed by maximizing the following equation

$$M_1^* = \frac{1}{(1+t^*)}\sum_{i=1}^{t^*+1}\ln\left(\Lambda_i^\bullet(\tau)\right).$$

Or, the MPS estimators $\widetilde{a}^{MPS}$ and $\widetilde{b}^{MPS}$ are produced by solving the following equations:

$$\frac{\partial M_1^*}{\partial a} = \frac{1}{t^*+1}\sum_{i=1}^{t^*+1}\frac{1}{\Lambda_i^\bullet(\tau)}\left\{1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a - \left\{1-\left[1-\left(1+y_{(i-1)}^{-1}\right)^{-b}\right]^a\right\}\right\}\pi_1\left(y_{(i)}|\tau\right) - \pi_1\left(y_{(i-1)}|\tau\right) = 0,$$

$$\frac{\partial M_1^*}{\partial b} = \frac{1}{t^*+1}\sum_{i=1}^{t^*+1}\frac{1}{\Lambda_i(\tau)}\left\{1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a - \left\{1-\left[1-\left(1+y_{(i-1)}^{-1}\right)^{-b}\right]^a\right\}\right\}\pi_2\left(y_{(i)}|\tau\right) - \pi_2\left(y_{(i-1)}|\tau\right) = 0,$$

where $\pi_1(.|\tau)$, and $\pi_2(.|\tau)$, share the same expression in Eqs. (26) and (27) by replacing $y_{(k:t^\bullet)}$ with $y_{(i)}$. Further, the MPS estimators $\widetilde{\gamma}_1^{MPS}$, $\widetilde{\gamma}_2^{MPS}$, $\widetilde{\gamma}_3^{MPS}$, and $\widetilde{\gamma}_4^{MPS}$ are obtained after inserting $\widetilde{a}^{MPS}$ and $\widetilde{b}^{MPS}$ in Eqs. (9), (11), (13), and (15), respectively.

5. Least squares & weighted least squares

In this section, we employ the LS estimation to determine $\widehat{a}^{LS}$ of a and $\widehat{b}^{LS}$ of b and the entropy estimators $\widehat{\gamma}_1^{LS}$, $\widehat{\gamma}_2^{LS}$, $\widehat{\gamma}_3^{LS}$, and $\widehat{\gamma}_4^{LS}$ of the IEP distribution. Furthermore we employ the WLS estimation to determine $\widehat{a}^{WS}$ of a and $\widehat{b}^{WS}$ of b and the entropy estimators $\widehat{\gamma}_1^{WS}$, $\widehat{\gamma}_2^{WS}$, $\widehat{\gamma}_3^{WS}$, and $\widehat{\gamma}_4^{WS}$ of the IEP distribution. All these estimators are created under RSS and SRS methods.

Suppose that $Y_{(1:t^*)}, \ldots, Y_{(t^*:t^*)}$, be the RSS with set size $t^* = t \times d$ where d is cycles count and t is the set size drawn from the IEP distribution with CDF (4). The LS and WLS estimators of unknown parameters of the IEP distribution are determined by minimizing the following error of sum squares.

$$l_3(\tau) = \sum_{k=1}^{t^*}\phi_k\left[1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a - \frac{k}{t^*+1}\right]^2.$$

The LS estimators $\widehat{a}^{LS}$ of a and $\widehat{b}^{LS}$ of b are provided for $\phi_k = 1$. While the WLS estimators $\widehat{a}^{WS}$ of a and $\widehat{b}^{WS}$ of b can be created when $\phi_k = \frac{(t^*+1)^2(t^*+2)}{k(t^*-k+1)}$. These estimates can also be obtained by solving the equations below using an iterative method:

$$\frac{\partial l_3(\tau)}{\partial a} = \sum_{k=1}^{t^*}\phi_k\left[1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a - \frac{k}{t^*+1}\right]\pi_1\left(y_{(k:t^\bullet)}|\tau\right) = 0,$$

$$\frac{\partial l_3(\tau)}{\partial b} = \sum_{k=1}^{t^*}\phi_k\left[1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a - \frac{k}{t^*+1}\right]\pi_2\left(y_{(k:t^\bullet)}|\tau\right) = 0,$$

where $\pi_1(.|\tau)$, and $\pi_2(.|\tau)$, are given in Eqs. (26) and (27). Further, following the same procedure presented by Jha et al. [57], the LS estimators $\widehat{\gamma}_1^{LS}$, $\widehat{\gamma}_2^{LS}$, $\widehat{\gamma}_3^{LS}$, and $\widehat{\gamma}_4^{LS}$ of the IEP distribution are obtained after inserting $\widehat{a}^{LS}$ and $\widehat{b}^{LS}$ in Eqs. (9), (11), (13), and (15), respectively. Similarly, WLS estimators $\widehat{\gamma}_1^{WS}$, $\widehat{\gamma}_2^{WS}$, $\widehat{\gamma}_3^{WS}$, and $\widehat{\gamma}_4^{WS}$ of the IEP distribution are obtained after inserting $\widehat{a}^{WS}$ and $\widehat{b}^{WS}$ in Eqs. (9), (11), (13), and (15), respectively.

After that, using the similar procedure discussed above, the LS and WLS estimators of parameters a and b are produced under SRS. To do so, let $y_{(1)}, \ldots, y_{(t^*)}$, be the RSS with set size t^* drawn from the IEP distribution with CDF (4). The LS and WLS estimators of unknown parameters of the IEP distribution are determined by minimizing the following error of sum squares.

$$l_3^\bullet(\tau) = \sum_{i=1}^{t^*}\phi_i\left[1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a - \frac{i}{t^*+1}\right]^2.$$

The LS estimators $\widetilde{a}^{LS}$ of a and $\widetilde{b}^{LS}$ of b are provided at for $\phi_i = 1$. While the WLS estimators $\widetilde{a}^{WS}$ of a and $\widetilde{b}^{WS}$ of b can be created when $\phi_i = \frac{(t^*+1)^2(t^*+2)}{i(t^*-i+1)}$. These estimates can also be obtained by solving the equations below using an iterative method:

$$\frac{\partial l_3^\bullet(\tau)}{\partial a} = \sum_{i=1}^{t^*}\phi_i\left[1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a - \frac{i}{t^*+1}\right]\pi_1\left(y_{(i)}|\tau\right) = 0,$$

$$\frac{\partial l_3^\bullet(\tau)}{\partial b} = \sum_{i=1}^{t^*}\phi_i\left[1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a - \frac{i}{t^*+1}\right]\pi_2\left(y_{(i)}|\tau\right) = 0,$$

where $\pi_1(.|\tau)$, and $\pi_2(.|\tau)$, are given in Eqs. (26) and (27). Further, following the same procedure presented by Jha et al. [57], the LS estimators $\widetilde{\gamma}_1^{LS}$, $\widetilde{\gamma}_2^{LS}$, $\widetilde{\gamma}_3^{LS}$, and $\widetilde{\gamma}_4^{LS}$ of the IEP distribution are obtained after inserting $\widetilde{a}^{LS}$ and $\widetilde{b}^{LS}$ in Eqs. (9), (11), (13), and (15), respectively. Similarly, the WLS estimators $\widetilde{\gamma}_1^{WS}$, $\widetilde{\gamma}_2^{WS}$, $\widetilde{\gamma}_3^{WS}$, and $\widetilde{\gamma}_4^{WS}$ of the IEP distribution are obtained after inserting $\widetilde{a}^{WS}$ and $\widetilde{b}^{WS}$ in Eqs. (9), (11), (13), and (15), respectively.

6. Estimators of minimum distance

The AD and the CVM estimation techniques for parameters a and b based on minimizing goodness-of-fit statistics are examined in this section. The difference between the data's empirical CDF and estimated CDF is measured by these statistics.

Let $Y_{(1:t^*)}, \ldots, Y_{(t^*:t^*)}$, be the RSS with set size $t^* = t \times d$ where d is cycles count and t is the set size drawn from the IEP distribution with CDF (4). The CVM estimators $\widehat{a}^{CM}$ of a and $\widehat{b}^{CM}$ of b are determined by minimizing the following function

$$l_4(\tau) = \frac{1}{12t^*} + \sum_{k=1}^{t^*}\left[1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a - \frac{2k-1}{2t^*}\right]^2. \tag{28}$$

The CVM estimators $\widehat{\gamma}_1^{\mathrm{CM}}$ of γ_1, $\widehat{\gamma}_2^{\mathrm{CM}}$ of γ_2, $\widehat{\gamma}_3^{\mathrm{CM}}$ of γ_3, and $\widehat{\gamma}_4^{\mathrm{CM}}$ of γ_4, of the IEP distribution are obtained after inserting $\widehat{a}^{\mathrm{CM}}$ and $\widehat{b}^{\mathrm{CM}}$ in Eqs. (9), (11), (13), and (15), respectively.

Using the similar procedure, let $y_{(1)}, \ldots, y_{(t^*)}$ be the SRS of size t^*, the CVM estimators $\widetilde{a}^{\mathrm{CM}}$ and $\widetilde{b}^{\mathrm{CM}}$ are obtained after minimizing Eq. (28) based on the SRS. Hence, following the same procedure presented by Jha et al. [57], the CVM estimators $\widetilde{\gamma}_1^{\mathrm{CM}}$ of γ_1, $\widetilde{\gamma}_2^{\mathrm{CM}}$ of γ_2, $\widetilde{\gamma}_3^{\mathrm{CM}}$ of γ_3, and $\widetilde{\gamma}_4^{\mathrm{CM}}$ of γ_4, of the IEP distribution are obtained after inserting $\widetilde{a}^{\mathrm{CM}}$ and $\widetilde{b}^{\mathrm{CM}}$ in Eqs. (9), (11), (13), and (15), respectively.

Let $Y_{(1:t^*)}, \ldots, Y_{(t^*:t^*)}$, be the RSS with set size $t^* = t \times d$ where d is cycles count and t is the set size drawn from the IEP distribution with CDF (4). The AD estimators $\widehat{a}^{\mathrm{AD}}$ of a and $\widehat{b}^{\mathrm{AD}}$ of b are presented by minimizing the following function

$$l_4^{\bullet}(\tau) = -t^* - \frac{1}{t^*} + \sum_{k=1}^{t^*}\left(2k-1\right)\left[\log\left\{1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a\right\} - \log\left\{\left[1-\left(1+y_{(t^*+k-1:t^*)}^{-1}\right)^{-b}\right]^a\right\}\right]. \tag{29}$$

The AD estimators $\widehat{\gamma}_1^{\mathrm{AD}}$ of γ_1, $\widehat{\gamma}_2^{\mathrm{AD}}$ of γ_2, $\widehat{\gamma}_3^{\mathrm{AD}}$ of γ_3, and $\widehat{\gamma}_4^{\mathrm{AD}}$ of γ_4, of the IEP distribution are obtained after inserting $\widehat{a}^{\mathrm{AD}}$ and $\widehat{b}^{\mathrm{AD}}$ in Eqs. (9), (11), (13), and (15), respectively.

Using the similar procedure, let $y_{(1)}, \ldots, y_{(t^*)}$ be the SRS of size t^*, the AD estimators $\widetilde{a}^{\mathrm{AD}}$ and $\widetilde{b}^{\mathrm{AD}}$ are obtained after minimizing Eq. (29) based on the SRS. Hence the AD estimators $\widetilde{\gamma}_1^{\mathrm{AD}}$ of γ_1, $\widetilde{\gamma}_2^{\mathrm{AD}}$ of γ_2, $\widetilde{\gamma}_3^{\mathrm{AD}}$ of γ_3, and $\widetilde{\gamma}_4^{\mathrm{AD}}$ of γ_4, of the IEP distribution are obtained after inserting $\widetilde{a}^{\mathrm{AD}}$ and $\widetilde{b}^{\mathrm{AD}}$ in Eqs. (9), (11), (13), and (15), respectively.

7. Estimators of Kolmogorov

The parameters a and b of the IEP distribution are determined using the KM method based on RSS and SRS. Furthermore, entropy estimators are also presented based on the two sampling schemes.

Let $Y_{(1:t^*)}, \ldots, Y_{(t^*:t^*)}$, be the RSS with set size $t^* = t \times d$ where d is cycles count and t is the set size drawn from the IEP distribution with CDF (4). The KM estimators $\widehat{a}^{\mathrm{KM}}$ of a and $\widehat{b}^{\mathrm{KM}}$ of b are presented by minimizing the following function

$$l_5(\tau) = \underset{1\le k\le t^*}{Max}\sum_{k=1}^{t^*}\left[\frac{k}{t^*} - \left\{1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a\right\}, \left\{1-\left[1-\left(1+y_{(k:t^*)}^{-1}\right)^{-b}\right]^a\right\} - \frac{k-1}{t^*}\right].$$

Also, the KM estimators $\widehat{\gamma}_1^{\mathrm{KM}}$, $\widehat{\gamma}_2^{\mathrm{KM}}$, $\widehat{\gamma}_3^{\mathrm{KM}}$, and $\widehat{\gamma}_4^{\mathrm{KM}}$ of the IEP distribution are obtained after inserting $\widehat{a}^{\mathrm{KM}}$ and $\widehat{b}^{\mathrm{KM}}$ in Eqs. (9), (11), (13), and (15), respectively.

Additionally, suppose that $Y_{(1)}, \ldots, Y_{(t^*)}$, be the SRS of size t^* drawn from the IEP distribution with CDF (4). The KM estimators $\widetilde{a}^{\mathrm{KM}}$ of a and $\widetilde{b}^{\mathrm{KM}}$ of b are presented by minimizing the following function

$$l_5^{\bullet}(\tau) = \underset{1\le i\le t^*}{Max}\sum_{i=1}^{t^*}\left[\frac{i}{t^*} - \left\{1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a\right\}, \left\{1-\left[1-\left(1+y_{(i)}^{-1}\right)^{-b}\right]^a\right\} - \frac{i-1}{t^*}\right].$$

Also, the KM estimators $\widetilde{\gamma}_1^{\mathrm{KM}}$, $\widetilde{\gamma}_2^{\mathrm{KM}}$, $\widetilde{\gamma}_3^{\mathrm{KM}}$, and $\widetilde{\gamma}_4^{\mathrm{KM}}$ of the IEP distribution are obtained after inserting $\widetilde{a}^{\mathrm{KM}}$ and $\widetilde{b}^{\mathrm{KM}}$ in Eqs. (9), (11), (13), and (15), respectively.

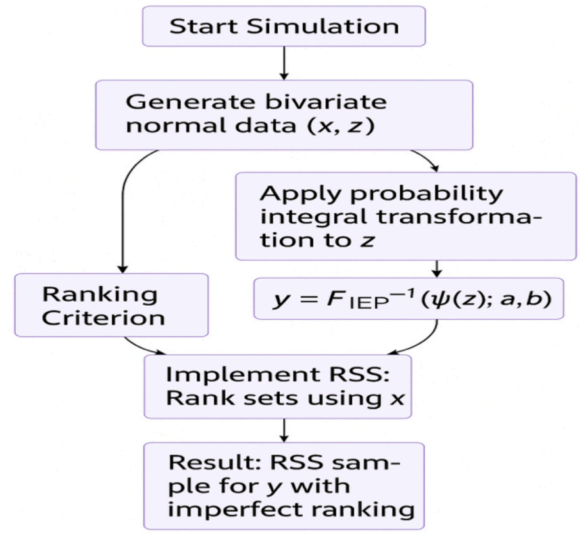

Fig. 2. Generating Imperfectly Ranked Samples using PIT and RSS.

8. Numerical evaluation

This section delves into examining the different estimation methods provided in this article. Through creating random datasets using the suggested model and ranking them, the purpose is to assess how well various approaches estimate the parameters. The suggested one is then determined by applying the estimate techniques. The following is how the simulation is executed:

- **Data Generation and Imperfect Ranking Mechanism:**

In order to generate data with imperfect ranking, we initially use a bivariate normal distribution to simulate data for the variable of interest y and a concomitant variable x with the following setting:

1. Generate correlated normals as:

$$\begin{pmatrix} x \\ z \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right),$$

where z is the latent normal variable and the correlation between x and z is denoted by ρ.

2. Apply probability integral transformation (PIT): To get the variable of interest y to follow the IEP (a, b) distribution, we transform the latent normal variable z using the PIT as $U = \psi(z) \sim$ Uniform $(0, 1)$.where, $\psi(z)$ is the CDF of the standard normal distribution applied to the variable z. This transforms z into a uniform value between 0 and 1. Then, we can get the variable of interest y as $y = F_{IEP}^{-1}[U; a, b]$, where, $F_{IEP}^{-1}[\ldots]$ is the inverse CDF (quantile function) of the IEP distribution with parameters a and b. Applying this to a uniform value transforms it into a value that follows the IEP distribution.
3. Perform Ranking: Next, the RSS design, which was explained in Section 1, was applied to y in a way that ranked the units in each set based on the values of the concomitant variable x. By adjusting ρ value, we were able to regulate the degree of imperfect ranking. For $\rho = 1$, the ranking situation is perfect, while for $\rho = 0$, the RSS design becomes the SRS design. A higher value of ρ indicates that the

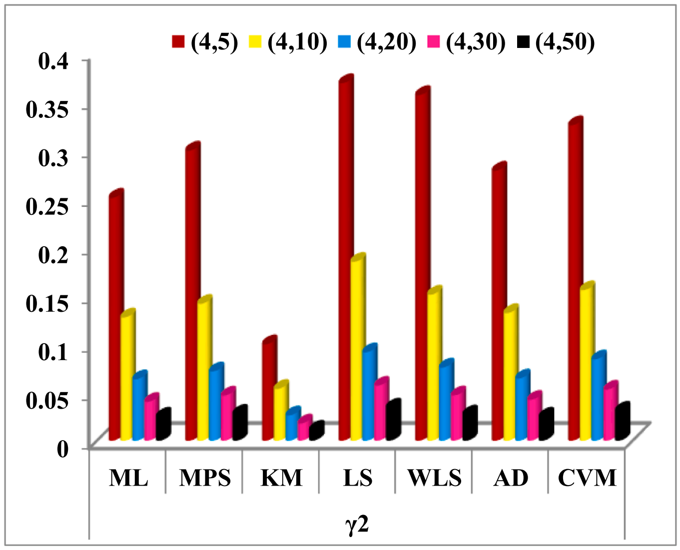

Fig. 3. MSEs of γ_2 estimates for all methods at $\rho = 1$ for Set 2.

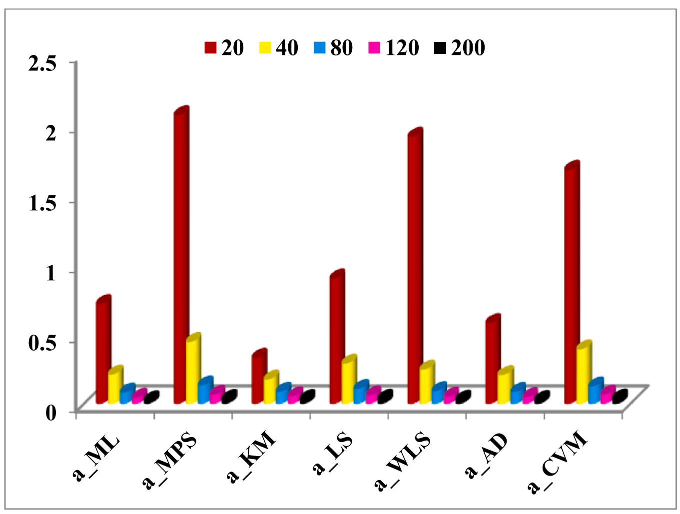

Fig. 4. MSEs of a estimates for all methods at $\rho = 0$ (SRS) for Set 3.

ranking mechanism is less imperfect. The chosen values of ρ are: $\rho = 1$ (perfect), 0.9 (imperfect), 0.7 (imperfect), and 0 (SRS). In addition, Fig. 2 depicts a detailed explanation of this work.

- We considered five different configurations to illustrate the effect of set size and different cycle numbers, respectively, as follows, $t = 4$ and $d = 5, 10, 20, 30,$ and 50, so the corresponding sample sizes are $t^* = t \times d = 20, 40, 80, 120,$ and 200.
- The true parameter values and the entropy parameter values are selected as $Set1 \equiv (a = 0.5, b = 0.8, \upsilon = 0.7)$, $Set2 \equiv (a = 2.5, b = 0.8, \upsilon = 0.7)$, $Set3 \equiv (a = 1.5, b = 2, \upsilon = 2)$ and $Set4 \equiv (a = 3.7, b = 1.5, \upsilon = 2)$, and the true values of γ_1, γ_2, γ_3, and γ_4 for Set1, Set2, Set3 and Set4, respectively, are (2.998, 4.859, 6.307 and 6.098), (0.570, 0.622, 0.807, and 0.646), (1.541, 0.786, 1.571, and 1.074) and (0.242,0.215, 0.429, and 0.227).
- A numerical technique is utilized to obtain the ML, MPS, KM, LS, WLS, AD, and CVM estimates for the parameters and the four mentioned entropy types above based on simulated samples.
- To assess the performance of the estimation methods, two measures are utilized, including an ABs and MSEs. We also report the REs of the estimates of the parameters and the four entropy types mentioned above, which are defined, respectively, as follows:

$$RE(a) = \frac{\mathrm{MSE_{SRS}}(\widetilde{a})}{\mathrm{MSE_{RSS}}(\widehat{a})}, \quad RE(b) = \frac{\mathrm{MSE_{SRS}}(\widetilde{b})}{\mathrm{MSE_{RSS}}(\widehat{b})}, \quad RE(\gamma_i) = \frac{\mathrm{MSE_{SRS}}(\widetilde{\gamma}_i)}{\mathrm{MSE_{RSS}}(\widehat{\gamma}_i)}, i = 1, 2, 3, 4$$

- The measures given above serve as objective standards for assessing the exactness and dependability of the calculated parameters. A comprehensive assessment of the estimating approaches' performance can be carried out by employing these evaluation criteria, providing important information about their efficiency and applicability for the precise model in discussion.
- Repeating this process multiple times, with 5000 iterations, we obtain a reliable assessment of the estimation techniques.
- The results are presented in Tables A.1–A.14 and Fig. 3–10, they provide a comprehensive overview of the outcomes obtained.
- According to Tables A.1 –A.14 and Fig. 3– 10, we remark the following:

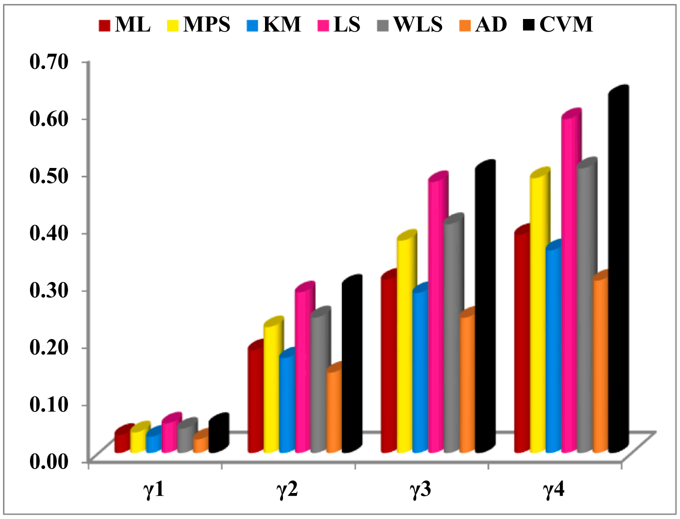

Fig. 5. MSEs of different entropies for all methods at ρ= 0.7, $(t, d) = (4, 5)$ for Set 1.

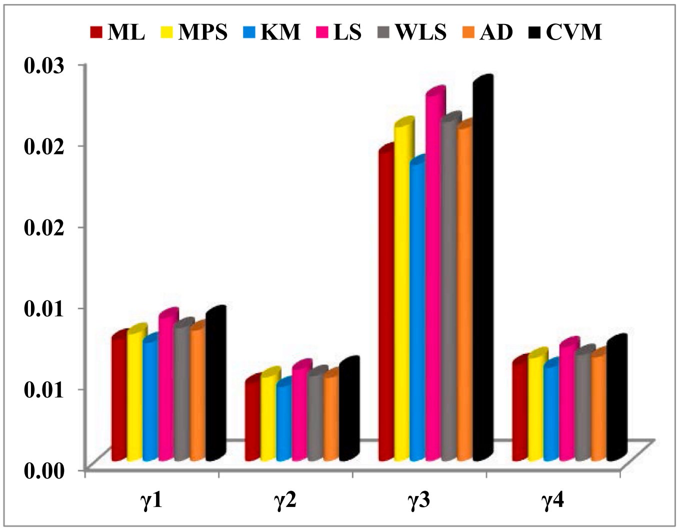

Fig. 6. MSEs of different entropies for all methods at ρ= 0 (SRS), $t^* = 120$ for Set 4.

1. Notably, our model estimates exhibit the consistency property for both SRS and RSS datasets. This property implies that as the sample size increases, the estimates converge to the true parameter values.
2. For both SRS and RSS (ρ=1, 0.9, and 0.7), the ABs and MSEs of all estimates tend to decrease as the t^* rises. Also, the ABs and MSEs of different entropy estimates decrease as t^* increases for SRS (ρ=0) and RSS (ρ=1, 0.9, and 0.7) (see Tables A.1–A.8 and Figs. 3 and 4).
3. In most cases, the KM method exhibits the smallest ABs and MSEs for SRS (ρ=0) and RSS (ρ=1, 0.9, and 0.7) datasets, indicating the quality of our estimates (see Tables A.1–A.8 and Figs. 3 and 4).
4. As expected, there is a touchable effect of the values of ρ on the behavior of the considered estimates, as increasing the values of ρ decreases the ABs and MSEs of all estimates and vice versa.
5. The estimates get more accurate as the sample size increases, indicating that they are asymptotically unbiased.
6. For fixed values of b and $\upsilon = 0.7$, the MSEs and ABs of a estimates increase and b estimates decrease for all methods of estimation and both sampling RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0), whereas the MSEs and ABs of γ_1, γ_2, γ_3 estimates increase while γ_4 estimates decrease for all methods of estimation and both sampling RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0) (see Tables A.1, A.2, A.5, and A.6).

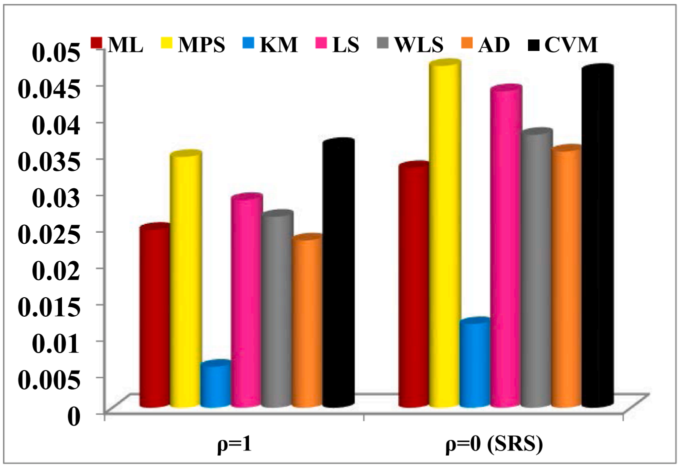

Fig. 7. MSEs of b estimates for all methods at $(t, d) = (4, 20)$, $t^* = 80$for Set 4.

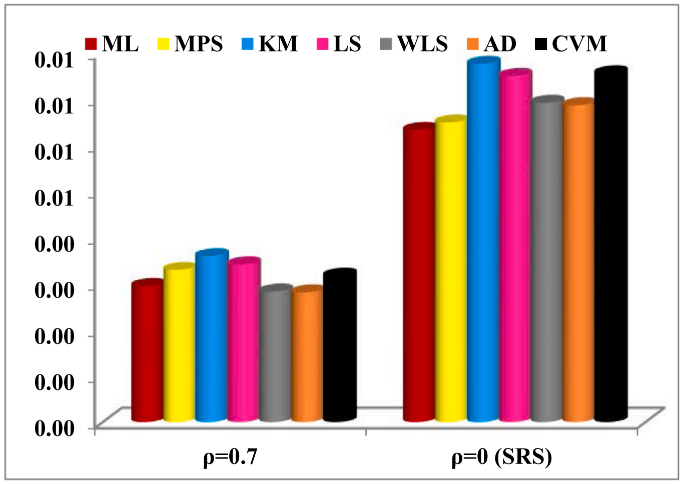

Fig. 8. MSEs of γ_1 estimates for all methods at $(t, d) = (4, 50)$, $t^* = 200$ for Set 3.

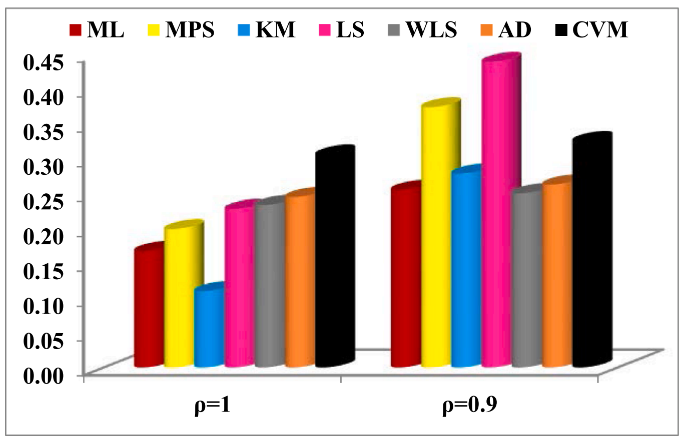

Fig. 9. MSEs of γ_3 estimates for all methods at $(t, d) = (4, 5)$, $t^* = 20$ for Set 1.

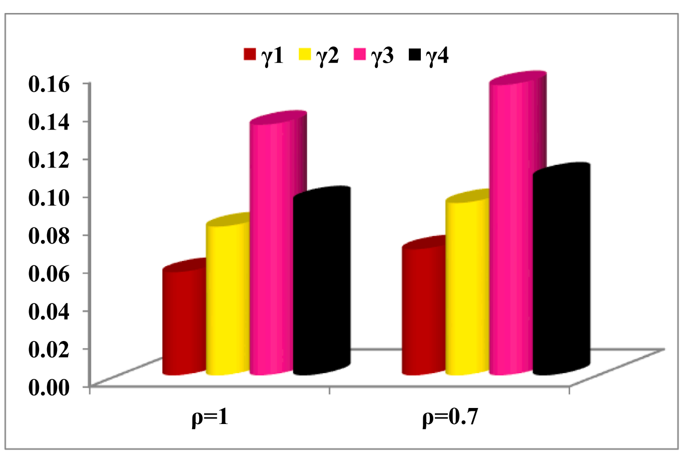

Fig. 10. MSEs of different entropies estimates for ML method at $(t, d) = (4, 10)$, $t^* = 40$ for Set 1.

7. The MSEs always decrease as the true values of γ_1, γ_2, γ_3 and γ_4 decrease and are less than one for RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0), indicating that the estimates are all consistent. As the true value of γ_1, γ_2, γ_3 and γ_4 increases and is greater than one, the MSEs are increasing for RSS (ρ=1, 0.9 and 0.7) and SRS (ρ=0).
8. The MSEs and ABs of a estimates increase and b estimates decrease when $\upsilon = 2$ at most methods of estimation and for RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0), whereas the MSEs and ABs of γ_2, γ_3, γ_4 estimates increase while γ_1 estimates decrease at most methods of estimation and for RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0) (see Tables A.3, A.4, A.7, and A.8).

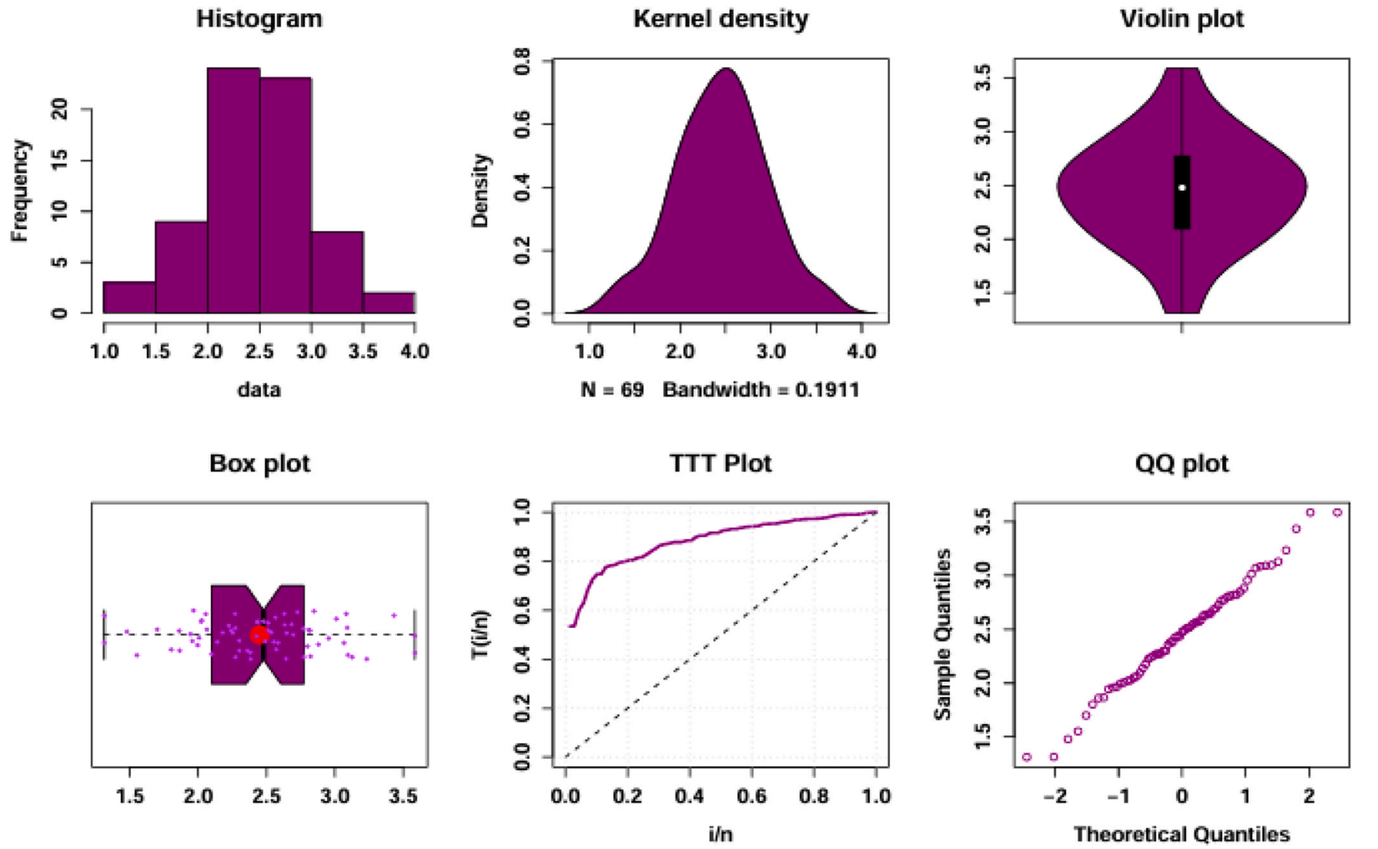

Fig. 11. Graphical representation for the considered real dataset.

Table 1
Applying descriptive statistics and visualization to the dataset.

Dataset	Sample size	Mean	Median	Variance	Skewness	Kurtosis	Range	Minimum	Maximum
	69	2.453	2.4780	0.2452	-0.0282	2.9407	2.273	1.3120	3.5850

9. When $\upsilon = 0.7$, for Sets 1 and 2, we notice that the estimate of γ_1 is superior to other entropy estimates for RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0). While the estimate of γ_2 is the best entropy estimates compared to others as the value of υ increases for RSS (ρ=1, 0.9, and 0.7) and SRS (ρ=0) (see Tables A.5 –A.8 and Figs. 5 and 6).
10. Figs. 7 and 8 show that the RSS for both perfect and imperfect ranking scenarios in most cases is preferable to the SRS (ρ=0)

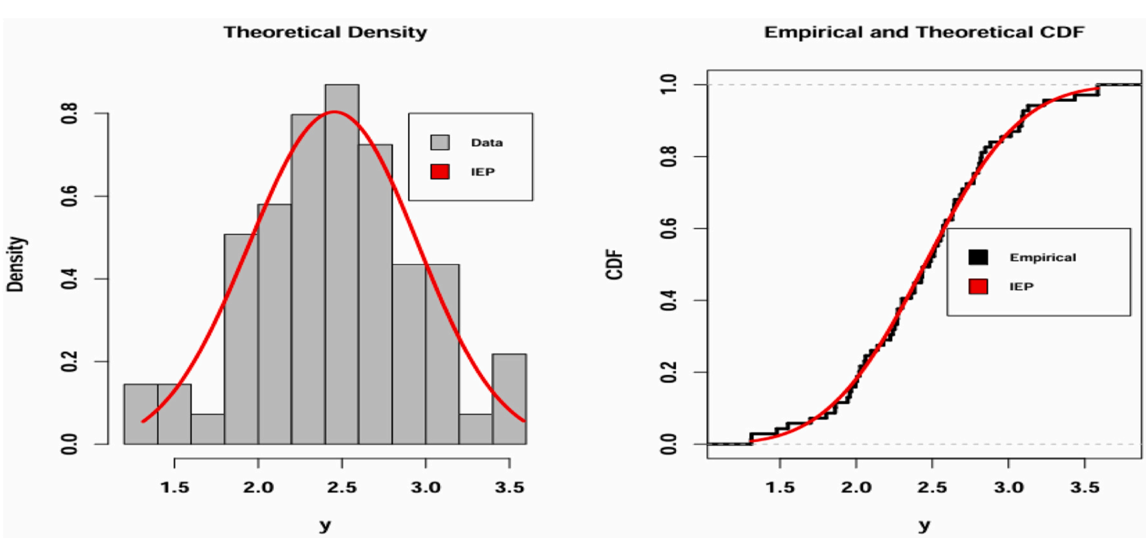

Fig. 12. Estimated PDF and CDF for the real dataset.

Table 2
Estimated parameter values of the IEP distribution calculated through different estimation approaches utilizing the RSS and SRS dataset.

ρ	Method	a	b
1	ML	56.931	12.055
	KM	1.485	4.131
	LS	25.779	10.616
	WLS	32.341	11.345
	AD	35.888	11.652
	CVM	21.093	10.056
0.9	ML	61.369	13.013
	KM	2.683	4.997
	LS	53.500	12.476
	WLS	38.905	11.509
	AD	36.285	11.766
	CVM	21.258	11.599
0.7	ML	63.141	14.168
	KM	2.847	5.781
	LS	64.047	12.022
	WLS	45.261	11.813
	AD	38.079	12.748
	CVM	46.702	11.760
0 (SRS)	ML	90.397	16.413
	KM	6.298	7.484
	LS	79.991	13.539
	WLS	47.410	14.408
	AD	42.379	14.124
	CVM	48.549	12.745

method at the parameter estimates and the four entropy estimates.

11. It is clear from Tables A.1–A.8 and Figs. 7 and 8 that the estimates obtained from the RSS (ρ=1, 0.9 and 0.7) datasets are more efficient than those obtained from the SRS (ρ=0) datasets. This implies that RSS (ρ=1, 0.9 and 0.7) is a more efficient sampling technique when it comes to generating estimates with a lower MSE.
12. For all RSS-based methods, the MSEs for the parameter estimates and different entropy estimates decrease as d and ρ increase.
13. In the case of perfect RSS (ρ=1) datasets, the estimates are more efficient than those in the imperfect RSS datasets, as can be shown from Tables A.1–A.8 and Figs. 9 and 10. This suggests that when it comes to generating estimates with a lower MSE, the perfect RSS is a more effective sampling method.
14. Larger values (greater than one) of RE indicate that the efficiency of the estimates under RSS outperforms the estimates under SRS in most cases.
15. As the cycle counts increase, the RE of parameter estimates and four entropy estimates increases in most situations (see Tables A.9–A.14).
16. A higher value of ρ ($\rho = 1$) tends to yield a higher RE for RSS, indicating better performance when the ranking is perfect, in contrast to an imperfect ranking.

9. Real data analysis

In this section, we discuss our results in detail and precisely select a real dataset to verify the feasibility of the suggested estimation methodologies. By thoroughly examining the real dataset, the intention was to demonstrate possible situations and uses for various estimating methodologies. This study clarifies the effective use of various estimate techniques, highlighting their applicability to applied research projects and well-informed decision-making. The data set represents tensile strength measures in GPA of 69 carbon fibers studied by Bader and Priest [58]. The data are as follows:

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382,2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Table 3
Estimated values of the various entropy types for the IEP distribution calculated through different estimation approaches utilizing the RSS and SRS dataset at υ = 0.7 and 2.

ρ	υ	Method	γ_1	γ_2	γ_3	γ_4
1	0.7	ML	1.099	1.302	1.690	1.404
		KM	2.335	3.382	4.389	4.013
		LS	1.128	1.342	1.742	1.450
		WLS	1.208	1.457	1.890	1.583
		AD	1.376	1.703	2.211	1.875
		CVM	1.201	1.446	1.877	1.571
	2	ML	0.850	0.573	1.145	0.693
		KM	1.690	0.816	1.631	1.141
		LS	0.879	0.585	1.170	0.712
		WLS	0.952	0.614	1.228	0.758
		AD	1.096	0.666	1.332	0.844
		CVM	0.946	0.612	1.223	0.753
0.9	0.7	ML	1.122	1.333	1.730	1.440
		KM	2.342	3.396	4.396	4.178
		LS	1.173	1.406	1.824	1.594
		WLS	1.239	1.500	1.947	1.634
		AD	1.393	1.730	2.245	1.897
		CVM	1.417	1.765	2.291	1.949
	2	ML	0.872	0.582	1.164	0.707
		KM	1.701	0.828	1.656	1.154
		LS	0.921	0.602	1.203	0.738
		WLS	0.978	0.624	1.248	0.774
		AD	1.175	0.671	1.342	0.853
		CVM	1.130	0.677	1.354	0.863
0.7	0.7	ML	1.225	1.480	1.921	1.611
		KM	2.896	4.612	5.986	5.737
		LS	1.422	1.774	2.302	1.959
		WLS	1.366	1.688	2.191	1.856
		AD	1.438	1.747	2.337	1.906
		CVM	1.484	1.869	2.426	2.074
	2	ML	0.974	0.622	1.245	0.771
		KM	2.001	0.865	1.729	1.264
		LS	1.146	0.682	1.364	0.872
		WLS	1.099	0.667	1.333	0.845
		AD	1.192	0.689	1.357	0.871
		CVM	1.197	0.698	1.396	0.901
0 (SRS)	0.7	ML	1.258	1.529	1.984	1.668
		KM	2.974	4.876	6.733	6.341
		LS	1.657	2.146	2.786	2.413
		WLS	1.415	1.763	2.289	1.947
		AD	1.450	1.817	2.358	2.011
		CVM	1.552	1.977	2.565	2.204
	2	ML	1.015	0.638	1.275	0.796
		KM	2.657	0.899	1.819	1.327
		LS	1.367	0.745	1.490	0.990
		WLS	1.160	0.687	1.373	0.880
		AD	1.212	0.696	1.393	0.898
		CVM	1.280	0.722	1.444	0.945

Descriptive measures and graphical representations in Fig. 11 are thoroughly examined as a part of the data scrutiny in Table 1.

These graphic depictions, which include violin plots, quantile-quantile (Q-Q) plots, histograms, kernel density plots, violin plots, total time on test (TTT) plots, and box plots, provide insightful looks into the characteristics and distribution patterns of the dataset. This data set appears to have a roughly symmetric distribution, but there might be slight skewness. There are no obvious outliers or irregularities in the data outliers, and it has an increasing HRF. The combined visualizations suggest that the data set is well-behaved and suitable for statistical modeling or further analysis.

Furthermore, the Kolmogorov-Smirnov (KS) test was used to evaluate how well the dataset fit an IEP distribution. With a p-value of 1.000 and a KS test of 0.0403, the results show that the selected distribution is appropriate for modeling the real dataset. Fig. 12 supports these findings.

Table 4
Goodness of fit measures by different estimation methods for dataset under RSS and SRS.

ρ	Method	KS	KS_ P-value	AD	AD_ P-value	CVM	CVM_ P-value
1	ML	0.293	0.452	0.398	0.384	0.074	0.323
	KM	0.380	0.224	1.231	0.183	0.244	0.131
	LS	0.284	0.430	0.482	0.411	0.088	0.355
	WLS	0.292	0.402	0.451	0.378	0.083	0.320
	AD	0.292	0.351	0.438	0.321	0.081	0.351
	CVM	0.287	0.403	0.514	0.381	0.094	0.323
0.9	ML	0.335	0.693	0.738	0.520	0.139	0.435
	KM	0.401	0.365	1.359	0.214	0.275	0.158
	LS	0.354	0.726	0.743	0.516	0.140	0.434
	WLS	0.348	0.696	0.792	0.479	0.149	0.401
	AD	0.381	0.695	0.871	0.426	0.166	0.364
	CVM	0.379	0.715	0.886	0.417	0.169	0.345
0.7	ML	0.355	0.728	0.941	0.845	0.177	0.748
	KM	0.436	0.406	1.475	0.254	0.303	0.197
	LS	0.361	0.756	0.895	0.757	0.165	0.666
	WLS	0.369	0.779	0.953	0.790	0.178	0.695
	AD	0.385	0.764	1.064	0.803	0.204	0.695
	CVM	0.399	0.721	0.947	0.724	0.177	0.636
0	ML	0.462	0.804	0.958	0.883	0.181	0.769
(SRS)	KM	0.437	0.519	1.997	0.354	0.390	0.293
	LS	0.400	0.765	0.907	0.845	0.197	0.782
	WLS	0.451	0.841	0.972	0.795	0.195	0.730
	AD	0.464	0.800	1.261	0.876	0.218	0.800
	CVM	0.481	0.737	0.974	0.774	0.182	0.744

Two sampling techniques SRS and RSS in both perfect and imperfect ranking were used to analyze the dataset in light of the theoretical conclusions and discussion previously provided, as ML, KM, LS, WLS, AD, and CVM, except the MPS approach to estimate the parameters and the above four entropy types for this data, since this dataset contains more equal values, then $\Lambda_k(\tau) = H(y_{(k:t^*)}) - H(y_{(k-1:t^*)}), k = i = 1, 2, \ldots, t^* \Lambda^{\bullet}_k(\tau) = H(y_{(k)}) - H(y_{(k-1)}), i = 1, 2, \ldots, t^*$, are equal to zero at most observations in RSS (ρ=1, 0.9, 0.7) and SRS (ρ=0). The estimates obtained from SRS and RSS in the case of perfect and imperfect ranking applied to the IEP distribution are displayed in Tables 2 and 3. Inferring from these data, we assume an SRS (ρ=0) of size 40, whereas the RSS (ρ=1, 0.9, 0.7) is assumed to have a small set size of $t = 4$ with a cycle count being $d = 10$. Significantly, the SRS and RSS in perfect and imperfect ranking techniques are compared using the same number of measurement units.

To accentuate the advantages of RSS in both perfect and imperfect ranking compared to SRS across the various estimation techniques, we conducted an evaluation utilizing multiple goodness of fit statistics, including the KS test, AD test, and CVM test, and their corresponding p-values. These statistical tests were employed to evaluate how well the data aligns with the model, offering insights into how effectively RSS, in the case of perfect and imperfect ranking, captures the underlying distribution compared to SRS. Table 4 presents a comparative analysis of the goodness of fit values between the SRS (ρ=0) and RSS (ρ=1, 0.9, 0.7) designs.

Through the use of goodness of fit metrics, a comparative analysis of the two designs makes it easier to evaluate how well each fits the dataset to the suggested model. Finding the sampling strategy and estimate methods that produce the best goodness of fit outcomes is the main goal of this comparison. It is routinely noticed that RSS in both perfect and imperfect ranking is superior to SRS across all estimates; also, the perfect ranking outperforms the imperfect ranking techniques. These results demonstrate the benefits of using RSS.

10. Concluding remarks

Entropy is a pivotal metric in information theory, quantifying the uncertainty or randomness of a system. Its significance extends across various disciplines. An effective sampling technique is the RSS, which is used when the variable of interest is reasonably easy to rank but expensive or challenging to measure. The precision and effectiveness of parameter estimation are improved by this technique. Regarding the importance of the IEP distribution in many fields, this paper investigates the estimation of its parameters and a number of entropy measures. The performance of these estimators is investigated in the frameworks of SRS and RSS in both perfect and imperfect ranking. The MPS, LS, KM, AD, WLS, ML, and CVM are the seven traditional estimate methods proposed for this purpose. The performance of estimates produced from SRS (ρ=0) and RSS (ρ=1, 0.9, 0.7) is assessed in a simulation study using the same number of measured units. Three criteria measures, such as AB, MSE, and RE, were developed for comparison. Based on numerical data, the KM technique seems to be useful in most situations for evaluating the estimated quality of RSS (ρ=1, 0.9, 0.7) and SRS (ρ=0). Because RSS, in the case of perfect and imperfect ranking, is more efficient at producing estimates with a lower MSE than SRS, it is a more effective sampling strategy. Also, the perfect ranking design does well than the imperfect ranking design. Actual data results that show the tensile strength measurements in GPA carbon fibers further strengthen the RSS in both perfect and imperfect ranking designs' superiority over the SRS design. The following points can be considered as future works:

- ➢ A comparative analysis with the traditional methods employed in this study may be possible by incorporating Bayesian techniques for parameter and entropy estimates. Bayesian techniques may increase estimation accuracy, especially for small sample sizes, especially when combined with informative priors.
- ➢ Expanding the research to include hybrid or adaptive censoring systems may shed light on entropy estimates in more intricate data-gathering situations, which are frequently encountered in reliability testing and survival analysis.
- ➢ Evaluate RSS's performance against other sophisticated sampling methods, like extreme ranked set sampling or double ranked set sampling; the most effective approach to entropy estimation under various circumstances may be found.

CRediT authorship contribution statement

Amal S. Hassan: Writing – review & editing, Writing – original draft, Valdiation, Resources, Methodology and formal analysis. **Tmader Alballa:** Writing – review & editing, Validation, Project administration, Investigation. **Rehab Alsultan:** Writing – review & editing, Software, Methodology, Data curation. **Etaf Alshawarbeh:** Writing – review & editing, Resources, Methodology, Data curation. **Rokaya Elmorsy Mohamed:** Writing – review & editing, Software, Resources, Methodology, Formal analysis. **Said G. Nassr:** Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis.

Declaration of Competing Interest

which declares that there is no conflict of interest

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Appendix

Table A.1
ABs and MSEs of IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set1 at $t = 4$

Set 1																	
t^*	d	Parameters	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	a	1	0.108	0.019	0.165	0.052	0.103	0.016	0.124	0.034	0.104	0.017	0.101	0.017	0.137	0.040
		b		0.231	0.103	0.370	0.293	0.182	0.063	0.269	0.135	0.262	0.120	0.267	0.102	0.298	0.192
40	10	a		0.061	0.006	0.089	0.014	0.064	0.008	0.073	0.008	0.064	0.006	0.069	0.007	0.077	0.010
		b		0.117	0.023	0.194	0.061	0.137	0.038	0.190	0.058	0.149	0.041	0.167	0.046	0.189	0.059
80	20	a		0.053	0.004	0.063	0.007	0.054	0.005	0.054	0.005	0.042	0.003	0.048	0.004	0.050	0.005
		b		0.116	0.023	0.143	0.035	0.104	0.022	0.137	0.028	0.107	0.019	0.103	0.016	0.117	0.021
120	30	a		0.041	0.003	0.041	0.003	0.039	0.003	0.042	0.003	0.036	0.002	0.035	0.002	0.040	0.003
		b		0.096	0.014	0.097	0.015	0.082	0.013	0.102	0.017	0.095	0.013	0.082	0.011	0.105	0.015
200	50	a		0.035	0.002	0.034	0.002	0.030	0.002	0.034	0.002	0.030	0.001	0.028	0.001	0.033	0.002
		b		0.075	0.009	0.080	0.010	0.069	0.009	0.094	0.014	0.075	0.009	0.068	0.008	0.065	0.007
20	5	a	0.9	0.118	0.027	0.167	0.065	0.109	0.026	0.126	0.045	0.106	0.018	0.106	0.022	0.122	0.031
		b		0.314	0.156	0.419	0.369	0.213	0.101	0.320	0.206	0.242	0.100	0.278	0.160	0.349	0.234
40	10	a		0.071	0.009	0.101	0.018	0.073	0.008	0.092	0.020	0.074	0.009	0.070	0.008	0.077	0.012
		b		0.199	0.069	0.253	0.117	0.173	0.056	0.221	0.097	0.186	0.054	0.169	0.051	0.199	0.088
80	20	a		0.046	0.004	0.060	0.006	0.053	0.005	0.060	0.005	0.050	0.004	0.050	0.004	0.055	0.005
		b		0.108	0.021	0.153	0.040	0.110	0.023	0.159	0.040	0.119	0.021	0.121	0.024	0.146	0.035
120	30	a		0.040	0.002	0.051	0.004	0.037	0.002	0.043	0.003	0.045	0.003	0.036	0.002	0.044	0.004
		b		0.099	0.016	0.128	0.027	0.086	0.014	0.106	0.019	0.120	0.021	0.097	0.017	0.120	0.026
200	50	a		0.029	0.001	0.028	0.001	0.033	0.002	0.037	0.002	0.028	0.001	0.032	0.002	0.032	0.002
		b		0.076	0.008	0.070	0.008	0.066	0.008	0.098	0.015	0.070	0.008	0.078	0.010	0.085	0.011
20	5	a	0.7	0.109	0.023	0.189	0.064	0.117	0.047	0.143	0.048	0.123	0.031	0.094	0.016	0.146	0.052
		b		0.268	0.139	0.474	0.415	0.237	0.186	0.300	0.185	0.273	0.126	0.253	0.130	0.425	0.436
40	10	a		0.068	0.008	0.091	0.014	0.074	0.010	0.088	0.014	0.085	0.012	0.069	0.007	0.089	0.016
		b		0.148	0.040	0.239	0.107	0.124	0.031	0.222	0.092	0.207	0.075	0.167	0.050	0.226	0.092
80	20	a		0.051	0.004	0.069	0.009	0.050	0.005	0.059	0.006	0.057	0.006	0.048	0.004	0.058	0.006
		b		0.119	0.025	0.167	0.052	0.110	0.020	0.139	0.033	0.148	0.036	0.128	0.026	0.157	0.042
120	30	a		0.041	0.003	0.052	0.004	0.035	0.002	0.045	0.003	0.051	0.005	0.045	0.003	0.048	0.004
		b		0.095	0.014	0.120	0.024	0.088	0.016	0.099	0.016	0.119	0.020	0.107	0.018	0.108	0.019
200	50	a		0.029	0.001	0.033	0.002	0.031	0.002	0.037	0.002	0.035	0.002	0.035	0.002	0.033	0.002
		b		0.060	0.007	0.082	0.011	0.076	0.011	0.084	0.013	0.090	0.014	0.086	0.012	0.084	0.014
20	5	a	0 (SRS)	0.127	0.043	0.204	0.080	0.110	0.018	0.143	0.056	0.126	0.035	0.128	0.031	0.162	0.063
		b		0.285	0.182	0.576	0.571	0.176	0.054	0.324	0.196	0.315	0.209	0.337	0.239	0.373	0.347
40	10	a		0.097	0.017	0.106	0.019	0.095	0.017	0.081	0.016	0.084	0.015	0.091	0.012	0.100	0.018
		b		0.197	0.076	0.269	0.197	0.174	0.07	0.245	0.099	0.195	0.065	0.199	0.072	0.24	0.098
80	20	a		0.054	0.005	0.065	0.013	0.059	0.006	0.071	0.008	0.055	0.005	0.052	0.004	0.072	0.010
		b		0.138	0.043	0.125	0.058	0.100	0.038	0.173	0.051	0.127	0.024	0.124	0.027	0.142	0.038
120	30	a		0.045	0.003	0.052	0.005	0.050	0.004	0.057	0.005	0.045	0.004	0.046	0.003	0.047	0.004
		b		0.098	0.018	0.099	0.037	0.096	0.016	0.108	0.029	0.126	0.023	0.098	0.023	0.132	0.029
200	50	a		0.035	0.002	0.042	0.003	0.039	0.003	0.040	0.003	0.033	0.002	0.036	0.002	0.044	0.003
		b		0.080	0.010	0.094	0.014	0.076	0.010	0.099	0.017	0.079	0.016	0.082	0.015	0.109	0.020

Table A.2
ABs and MSEs of IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 2 at $t = 4$

Set 2																	
t^*	d	Parameters	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	a	1	0.869	0.553	0.476	0.819	0.092	0.042	0.992	0.695	0.820	0.673	0.795	0.957	0.102	0.610
		b		0.150	0.042	0.212	0.084	0.068	0.007	0.168	0.046	0.167	0.045	0.171	0.039	0.179	0.060
40	10	a		0.425	0.306	0.719	0.960	0.118	0.073	0.582	0.582	0.473	0.397	0.519	0.434	0.612	0.686
		b		0.076	0.010	0.123	0.023	0.052	0.005	0.121	0.023	0.098	0.016	0.105	0.018	0.118	0.022
80	20	a		0.398	0.265	0.490	0.400	0.098	0.035	0.418	0.286	0.324	0.199	0.341	0.192	0.369	0.260
		b		0.074	0.009	0.089	0.013	0.038	0.002	0.086	0.011	0.069	0.007	0.065	0.007	0.075	0.009
120	30	a		0.290	0.133	0.327	0.192	0.074	0.033	0.318	0.181	0.272	0.112	0.259	0.105	0.312	0.155
		b		0.059	0.005	0.065	0.007	0.032	0.002	0.065	0.007	0.060	0.005	0.053	0.005	0.067	0.006
200	50	a		0.262	0.110	0.248	0.097	0.051	0.009	0.270	0.122	0.226	0.082	0.204	0.075	0.232	0.090
		b		0.051	0.004	0.049	0.004	0.026	0.001	0.060	0.006	0.048	0.004	0.044	0.003	0.043	0.003
20	5	a	0.9	0.940	0.698	0.623	0.857	0.173	0.212	1.016	0.711	0.840	0.695	0.799	0.965	0.940	0.698
		b		0.192	0.056	0.246	0.099	0.092	0.014	0.175	0.050	0.168	0.047	0.163	0.043	0.192	0.056
40	10	a		0.585	0.707	0.878	0.972	0.127	0.103	0.768	1.826	0.610	0.632	0.543	0.529	0.585	0.707
		b		0.118	0.023	0.146	0.035	0.067	0.007	0.132	0.033	0.125	0.023	0.110	0.022	0.118	0.023
80	20	a		0.340	0.204	0.459	0.381	0.076	0.024	0.435	0.296	0.379	0.227	0.364	0.210	0.340	0.204
		b		0.071	0.008	0.087	0.013	0.042	0.003	0.100	0.015	0.078	0.010	0.075	0.009	0.071	0.008
120	30	a		0.282	0.136	0.409	0.301	0.071	0.026	0.325	0.183	0.301	0.146	0.259	0.126	0.282	0.136
		b		0.059	0.006	0.081	0.010	0.037	0.002	0.073	0.008	0.064	0.006	0.060	0.006	0.059	0.006
200	50	a		0.216	0.071	0.218	0.083	0.065	0.015	0.284	0.125	0.196	0.063	0.201	0.073	0.216	0.071

(continued on next page)

Table A.2 (*continued*)

Set 2																	
		b		0.047	0.003	0.048	0.004	0.028	0.001	0.062	0.006	0.043	0.003	0.042	0.003	0.047	0.003
20	5	*a*	0.7	0.973	0.754	0.601	0.860	0.165	0.103	1.160	0.729	0.900	0.707	0.690	0.977	0.973	0.754
		b		0.167	0.058	0.260	0.113	0.095	0.016	0.199	0.070	0.179	0.059	0.135	0.047	0.167	0.058
40	10	*a*		0.590	0.703	0.752	0.994	0.116	0.066	0.633	0.665	0.655	0.791	0.523	0.463	0.590	0.703
		b		0.110	0.021	0.145	0.035	0.070	0.008	0.121	0.025	0.134	0.027	0.113	0.020	0.110	0.021
80	20	*a*		0.354	0.226	0.570	0.636	0.075	0.022	0.444	0.370	0.450	0.359	0.318	0.179	0.354	0.226
		b		0.075	0.009	0.100	0.017	0.052	0.004	0.088	0.013	0.094	0.014	0.080	0.010	0.075	0.009
120	30	*a*		0.279	0.126	0.414	0.280	0.058	0.013	0.331	0.160	0.354	0.222	0.304	0.146	0.279	0.126
		b		0.057	0.006	0.084	0.011	0.037	0.002	0.066	0.007	0.070	0.008	0.065	0.006	0.057	0.006
200	50	*a*		0.208	0.079	0.238	0.096	0.048	0.008	0.276	0.132	0.257	0.100	0.245	0.092	0.208	0.079
		b		0.046	0.004	0.051	0.004	0.035	0.002	0.056	0.005	0.055	0.005	0.052	0.004	0.046	0.004
20	5	*a*	0	0.996	0.855	0.811	0.986	0.181	0.166	1.175	0.752	0.968	0.741	0.754	1.000	1.408	0.699
		b	(SRS)	0.170	0.060	0.322	0.165	0.106	0.018	0.203	0.089	0.194	0.062	0.196	0.072	0.265	0.092
40	10	*a*		0.702	0.897	0.799	1.075	0.125	0.073	0.636	0.810	0.673	0.835	0.639	0.737	0.780	0.970
		b		0.124	0.025	0.149	0.036	0.075	0.009	0.134	0.030	0.142	0.029	0.126	0.026	0.145	0.043
80	20	*a*		0.387	0.268	0.650	0.755	0.073	0.032	0.547	0.519	0.486	0.425	0.354	0.212	0.495	0.546
		b		0.090	0.013	0.286	0.021	0.062	0.006	0.106	0.019	0.096	0.020	0.089	0.010	0.117	0.019
120	30	*a*		0.286	0.133	0.449	0.320	0.083	0.030	0.403	0.238	0.394	0.251	0.392	0.149	0.378	0.295
		b		0.060	0.007	0.087	0.019	0.047	0.003	0.080	0.010	0.089	0.010	0.075	0.007	0.074	0.009
200	50	*a*		0.241	0.096	0.301	0.156	0.055	0.010	0.295	0.153	0.287	0.293	0.269	0.095	0.324	0.178
		b		0.047	0.005	0.062	0.006	0.044	0.003	0.060	0.006	0.058	0.006	0.053	0.004	0.068	0.008

Table A.3
ABs and MSEs of IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 3 at $t = 4$

Set 3																	
t^*	*d*	Parameters	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	*a*	1	0.440	0.359	0.721	1.210	0.361	0.201	0.523	0.887	0.420	0.313	0.411	0.278	0.570	0.766
		b		0.412	0.325	0.599	0.680	0.352	0.198	0.469	0.380	0.463	0.348	0.474	0.304	0.505	0.483
40	10	*a*		0.226	0.082	0.366	0.244	0.247	0.129	0.300	0.148	0.247	0.104	0.271	0.114	0.314	0.174
		b		0.210	0.076	0.339	0.178	0.253	0.113	0.335	0.175	0.270	0.126	0.292	0.137	0.329	0.172
80	20	*a*		0.209	0.071	0.254	0.107	0.220	0.091	0.217	0.076	0.168	0.053	0.182	0.053	0.194	0.071
		b		0.206	0.071	0.251	0.100	0.203	0.077	0.239	0.084	0.190	0.058	0.183	0.051	0.209	0.066
120	30	*a*		0.154	0.037	0.167	0.049	0.175	0.059	0.166	0.048	0.142	0.031	0.136	0.028	0.161	0.042
		b		0.165	0.041	0.179	0.050	0.184	0.055	0.181	0.053	0.166	0.040	0.148	0.035	0.186	0.048
200	50	*a*		0.137	0.030	0.131	0.026	0.112	0.021	0.139	0.033	0.118	0.022	0.107	0.021	0.124	0.025
		b		0.140	0.030	0.138	0.031	0.120	0.023	0.167	0.043	0.134	0.029	0.121	0.024	0.118	0.023
20	5	*a*	0.9	0.524	0.548	0.795	1.552	0.386	0.349	0.515	0.706	0.420	0.478	0.415	0.429	0.586	1.578
		b		0.537	0.445	0.702	0.850	0.429	0.326	0.507	0.452	0.450	0.354	0.467	0.369	0.594	0.684
40	10	*a*		0.302	0.195	0.436	0.342	0.289	0.148	0.407	0.521	0.293	0.128	0.282	0.136	0.321	0.302
		b		0.328	0.196	0.395	0.277	0.322	0.184	0.368	0.269	0.321	0.145	0.311	0.167	0.353	0.255
80	20	*a*		0.178	0.058	0.238	0.102	0.191	0.062	0.235	0.085	0.209	0.067	0.193	0.059	0.223	0.083
		b		0.204	0.072	0.239	0.103	0.217	0.078	0.284	0.117	0.230	0.078	0.206	0.068	0.251	0.099
120	30	*a*		0.151	0.036	0.208	0.076	0.176	0.056	0.165	0.046	0.159	0.042	0.140	0.035	0.177	0.057
		b		0.167	0.046	0.214	0.072	0.190	0.065	0.194	0.056	0.176	0.044	0.160	0.047	0.215	0.075
200	50	*a*		0.113	0.020	0.115	0.023	0.131	0.029	0.145	0.034	0.105	0.018	0.107	0.021	0.134	0.033
		b		0.123	0.025	0.136	0.029	0.130	0.031	0.164	0.045	0.122	0.025	0.118	0.025	0.153	0.035
20	5	*a*	0.7	0.438	0.391	0.760	1.180	0.351	0.289	0.609	1.004	0.463	0.414	0.368	0.343	0.589	0.835
		b		0.467	0.370	0.709	0.884	0.386	0.305	0.569	0.585	0.518	0.473	0.398	0.285	0.644	0.804
40	10	*a*		0.300	0.171	0.399	0.267	0.294	0.152	0.349	0.200	0.341	0.203	0.280	0.125	0.395	0.430
		b		0.293	0.149	0.416	0.305	0.281	0.143	0.343	0.208	0.369	0.209	0.324	0.171	0.374	0.256
80	20	*a*		0.187	0.060	0.289	0.162	0.184	0.052	0.235	0.109	0.236	0.095	0.171	0.053	0.252	0.100
		b		0.198	0.066	0.276	0.133	0.190	0.062	0.251	0.102	0.263	0.105	0.227	0.078	0.293	0.137
120	30	*a*		0.145	0.033	0.209	0.075	0.154	0.043	0.181	0.047	0.186	0.061	0.166	0.043	0.196	0.077
		b		0.156	0.040	0.228	0.079	0.181	0.059	0.178	0.048	0.193	0.058	0.191	0.052	0.201	0.066
200	50	*a*		0.110	0.020	0.127	0.028	0.129	0.030	0.146	0.036	0.134	0.027	0.132	0.025	0.131	0.034
		b		0.124	0.026	0.138	0.031	0.146	0.035	0.160	0.041	0.153	0.036	0.149	0.035	0.146	0.044
20	5	*a*	0	0.502	0.813	0.951	1.862	0.299	0.159	0.565	0.600	0.496	0.566	0.528	0.684	0.704	1.679
		b	(SRS)	0.478	0.465	0.916	1.345	0.348	0.199	0.565	0.460	0.541	0.495	0.554	0.585	0.609	0.775
40	10	*a*		0.370	0.246	0.414	0.282	0.339	0.224	0.325	0.204	0.305	0.168	0.338	0.192	0.392	0.488
		b		0.346	0.197	0.416	0.282	0.302	0.166	0.374	0.235	0.340	0.188	0.352	0.207	0.407	0.263
80	20	*a*		0.203	0.072	0.239	0.100	0.203	0.077	0.284	0.135	0.205	0.064	0.190	0.060	0.265	0.147
		b		0.251	0.104	0.219	0.083	0.202	0.066	0.296	0.146	0.225	0.075	0.220	0.081	0.243	0.107
120	30	*a*		0.153	0.040	0.187	0.063	0.169	0.050	0.211	0.066	0.159	0.044	0.160	0.042	0.173	0.056
		b		0.154	0.039	0.198	0.068	0.179	0.053	0.224	0.077	0.165	0.046	0.180	0.048	0.204	0.065
200	50	*a*		0.130	0.027	0.159	0.043	0.136	0.032	0.143	0.036	0.121	0.022	0.129	0.026	0.170	0.049
		b		0.131	0.027	0.173	0.045	0.133	0.030	0.147	0.034	0.140	0.030	0.147	0.034	0.189	0.059

Table A.4
ABs and MSEs of IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 4 at $t = 4$

Set 4																	
t^*	d	Parameters	ρ	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	a	1	0.876	0.523	0.787	0.841	0.133	0.131	0.384	0.724	0.373	0.807	0.322	0.455	0.698	0.443
		b		0.265	0.132	0.371	0.253	0.114	0.020	0.286	0.124	0.294	0.139	0.301	0.120	0.300	0.152
40	10	a		0.694	0.457	0.610	0.774	0.043	0.007	0.268	0.674	0.278	0.516	0.854	0.318	0.020	0.382
		b		0.135	0.032	0.217	0.072	0.067	0.007	0.213	0.070	0.173	0.051	0.184	0.054	0.208	0.069
80	20	a		0.655	0.331	0.316	0.516	0.034	0.003	0.192	0.591	0.236	0.451	0.555	0.215	0.238	0.313
		b		0.130	0.028	0.157	0.039	0.058	0.006	0.151	0.033	0.121	0.023	0.115	0.020	0.133	0.027
120	30	a		0.471	0.335	0.247	0.510	0.030	0.002	0.124	0.503	0.147	0.301	0.425	0.206	0.219	0.226
		b		0.102	0.016	0.115	0.021	0.042	0.003	0.115	0.022	0.105	0.016	0.094	0.014	0.118	0.019
200	50	a		0.431	0.300	0.208	0.364	0.028	0.019	0.118	0.336	0.112	0.224	0.334	0.202	0.177	0.241
		b		0.089	0.012	0.087	0.012	0.041	0.003	0.106	0.017	0.084	0.011	0.077	0.010	0.075	0.010
20	5	a	0.9	0.908	0.654	0.917	0.897	0.162	0.149	0.423	0.789	0.403	0.914	0.375	0.576	0.698	0.581
		b		0.321	0.166	0.421	0.293	0.118	0.022	0.288	0.126	0.278	0.124	0.284	0.128	0.319	0.177
40	10	a		0.901	0.482	0.382	0.829	0.233	0.112	0.417	0.737	0.344	0.719	0.341	0.324	0.345	0.454
		b		0.196	0.076	0.228	0.092	0.079	0.010	0.204	0.075	0.203	0.064	0.184	0.053	0.219	0.082
80	20	a		0.765	0.454	0.366	0.605	0.163	0.023	0.342	0.665	0.256	0.717	0.282	0.311	0.317	0.392
		b		0.127	0.026	0.150	0.038	0.058	0.006	0.177	0.049	0.144	0.032	0.129	0.024	0.162	0.041
120	30	a		0.447	0.364	0.281	0.536	0.150	0.010	0.302	0.424	0.189	0.389	0.253	0.306	0.302	0.359
		b		0.102	0.018	0.138	0.029	0.051	0.004	0.112	0.019	0.104	0.018	0.102	0.019	0.130	0.029
200	50	a		0.361	0.205	0.178	0.258	0.046	0.020	0.300	0.319	0.126	0.379	0.240	0.302	0.291	0.321
		b		0.083	0.011	0.088	0.013	0.040	0.003	0.103	0.017	0.077	0.009	0.078	0.011	0.089	0.013
20	5	a	0.7	0.977	0.465	0.981	0.920	0.194	0.179	0.464	0.192	0.610	0.734	0.384	0.664	0.828	0.652
		b		0.282	0.136	0.406	0.305	0.133	0.029	0.339	0.189	0.326	0.192	0.250	0.110	0.373	0.228
40	10	a		0.963	0.498	0.977	0.911	0.188	0.156	0.438	0.171	0.550	0.728	0.359	0.600	0.817	0.593
		b		0.203	0.071	0.257	0.111	0.107	0.020	0.206	0.069	0.229	0.086	0.202	0.061	0.238	0.100
80	20	a		0.838	0.469	0.925	0.910	0.172	0.141	0.422	0.170	0.535	0.714	0.342	0.599	0.760	0.588
		b		0.119	0.024	0.169	0.050	0.079	0.009	0.156	0.037	0.154	0.040	0.133	0.030	0.178	0.046
120	30	a		0.734	0.406	0.878	0.884	0.168	0.140	0.412	0.168	0.501	0.656	0.319	0.577	0.609	0.529
		b		0.099	0.016	0.146	0.032	0.066	0.008	0.119	0.020	0.127	0.024	0.116	0.020	0.127	0.027
200	50	a		0.431	0.398	0.801	0.845	0.147	0.100	0.389	0.152	0.403	0.640	0.302	0.543	0.426	0.437
		b		0.078	0.010	0.088	0.014	0.050	0.004	0.099	0.016	0.093	0.013	0.091	0.013	0.099	0.017
20	5	a	0	0.994	0.524	0.999	0.986	0.347	0.234	0.610	0.819	0.699	0.821	0.499	0.704	0.858	0.798
		b	(SRS)	0.298	0.176	0.561	0.496	0.159	0.039	0.352	0.176	0.342	0.189	0.341	0.216	0.352	0.221
40	10	a		0.964	0.511	0.994	0.956	0.236	0.221	0.588	0.751	0.633	0.800	0.362	0.622	0.839	0.631
		b		0.217	0.077	0.261	0.109	0.119	0.022	0.237	0.093	0.215	0.074	0.222	0.080	0.255	0.102
80	20	a		0.843	0.498	0.936	0.926	0.193	0.219	0.505	0.616	0.629	0.761	0.355	0.612	0.804	0.616
		b		0.158	0.041	0.138	0.033	0.080	0.009	0.186	0.057	0.142	0.030	0.139	0.032	0.153	0.042
120	30	a		0.778	0.443	0.918	0.894	0.173	0.184	0.475	0.179	0.573	0.694	0.340	0.578	0.739	0.561
		b		0.096	0.015	0.124	0.026	0.071	0.008	0.142	0.030	0.104	0.018	0.115	0.019	0.130	0.026
200	50	a		0.590	0.358	0.893	0.885	0.153	0.112	0.427	0.162	0.471	0.653	0.326	0.555	0.531	0.485
		b		0.084	0.011	0.110	0.018	0.057	0.005	0.092	0.014	0.089	0.012	0.093	0.013	0.119	0.023

Table A.5
ABs and MSEs of the different entropy measures for IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set1 at $t = 4$

Set 1																	
t^*	d	ρ	Entropy-Type	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	1	γ_1	0.099	0.017	0.110	0.020	0.083	0.011	0.120	0.024	0.124	0.024	0.131	0.025	0.125	0.034
			γ_2	0.239	0.099	0.270	0.118	0.202	0.065	0.288	0.135	0.299	0.138	0.315	0.145	0.300	0.181
			γ_3	0.311	0.167	0.350	0.198	0.263	0.109	0.373	0.227	0.388	0.233	0.409	0.244	0.389	0.305
			γ_4	0.350	0.211	0.396	0.254	0.297	0.139	0.419	0.285	0.437	0.294	0.460	0.307	0.436	0.375
40	10		γ_1	0.062	0.006	0.070	0.008	0.070	0.008	0.084	0.010	0.071	0.008	0.071	0.008	0.078	0.010
			γ_2	0.151	0.037	0.173	0.046	0.171	0.047	0.205	0.060	0.172	0.046	0.173	0.046	0.191	0.057
			γ_3	0.197	0.062	0.224	0.078	0.222	0.080	0.266	0.101	0.224	0.078	0.224	0.077	0.248	0.096
			γ_4	0.222	0.080	0.254	0.100	0.250	0.101	0.301	0.129	0.253	0.099	0.253	0.098	0.280	0.123
80	20		γ_1	0.051	0.004	0.052	0.004	0.046	0.004	0.054	0.005	0.048	0.004	0.045	0.003	0.055	0.005
			γ_2	0.126	0.024	0.129	0.025	0.113	0.022	0.132	0.027	0.117	0.022	0.111	0.019	0.134	0.027
			γ_3	0.163	0.041	0.167	0.042	0.147	0.037	0.172	0.046	0.152	0.037	0.144	0.032	0.174	0.046
			γ_4	0.185	0.052	0.190	0.054	0.166	0.047	0.194	0.059	0.172	0.047	0.163	0.041	0.196	0.058
120	30		γ_1	0.043	0.003	0.036	0.002	0.037	0.002	0.043	0.003	0.044	0.003	0.039	0.002	0.043	0.003
			γ_2	0.104	0.018	0.088	0.012	0.090	0.013	0.104	0.017	0.108	0.017	0.096	0.015	0.106	0.016
			γ_3	0.135	0.030	0.114	0.020	0.117	0.022	0.136	0.029	0.141	0.029	0.124	0.025	0.137	0.027
			γ_4	0.153	0.038	0.130	0.026	0.132	0.028	0.154	0.038	0.159	0.037	0.141	0.032	0.156	0.035
200	50		γ_1	0.032	0.002	0.033	0.002	0.031	0.002	0.042	0.003	0.031	0.002	0.030	0.002	0.028	0.001
			γ_2	0.078	0.009	0.081	0.010	0.077	0.010	0.103	0.016	0.076	0.009	0.073	0.009	0.069	0.008
			γ_3	0.101	0.015	0.105	0.016	0.100	0.017	0.133	0.026	0.098	0.016	0.095	0.015	0.089	0.013
			γ_4	0.114	0.020	0.119	0.021	0.113	0.021	0.151	0.034	0.111	0.020	0.108	0.020	0.101	0.016
20	5	0.9	γ_1	0.131	0.026	0.140	0.039	0.129	0.030	0.159	0.048	0.125	0.026	0.133	0.027	0.142	0.032
			γ_2	0.319	0.151	0.343	0.222	0.310	0.165	0.381	0.261	0.302	0.148	0.322	0.156	0.348	0.193

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Table A.5 (*continued*)

Set 1																	
			γ_3	0.414	0.255	0.445	0.374	0.403	0.278	0.495	0.439	0.391	0.250	0.418	0.262	0.452	0.325
			γ_4	0.468	0.324	0.503	0.470	0.452	0.346	0.556	0.542	0.439	0.312	0.471	0.332	0.512	0.416
40	10		γ_1	0.081	0.010	0.093	0.013	0.084	0.011	0.097	0.014	0.091	0.013	0.086	0.012	0.094	0.015
			γ_2	0.200	0.060	0.230	0.078	0.205	0.067	0.236	0.084	0.221	0.078	0.210	0.068	0.229	0.087
			γ_3	0.259	0.102	0.299	0.131	0.266	0.114	0.307	0.141	0.287	0.131	0.272	0.114	0.297	0.146
			γ_4	0.294	0.131	0.339	0.169	0.301	0.146	0.346	0.178	0.324	0.166	0.307	0.145	0.335	0.186
80	20		γ_1	0.049	0.004	0.068	0.007	0.056	0.005	0.083	0.010	0.058	0.005	0.058	0.006	0.073	0.008
			γ_2	0.120	0.024	0.168	0.042	0.137	0.031	0.204	0.058	0.141	0.030	0.142	0.034	0.178	0.048
			γ_3	0.156	0.040	0.217	0.070	0.178	0.053	0.264	0.098	0.183	0.050	0.185	0.058	0.231	0.082
			γ_4	0.176	0.051	0.246	0.090	0.202	0.068	0.299	0.126	0.207	0.064	0.209	0.073	0.261	0.104
120	30		γ_1	0.048	0.004	0.050	0.004	0.042	0.003	0.054	0.005	0.053	0.004	0.045	0.003	0.051	0.004
			γ_2	0.119	0.021	0.124	0.025	0.104	0.018	0.132	0.029	0.131	0.025	0.112	0.019	0.126	0.027
			γ_3	0.154	0.036	0.160	0.042	0.135	0.031	0.172	0.048	0.170	0.041	0.145	0.032	0.164	0.045
			γ_4	0.174	0.046	0.182	0.054	0.153	0.039	0.194	0.062	0.192	0.053	0.164	0.041	0.186	0.058
200	50		γ_1	0.036	0.002	0.033	0.002	0.037	0.002	0.042	0.003	0.035	0.002	0.037	0.002	0.040	0.002
			γ_2	0.089	0.013	0.080	0.009	0.090	0.013	0.103	0.018	0.085	0.011	0.092	0.013	0.099	0.014
			γ_3	0.115	0.022	0.104	0.016	0.116	0.022	0.133	0.030	0.110	0.019	0.119	0.022	0.128	0.024
			γ_4	0.130	0.028	0.118	0.020	0.132	0.028	0.151	0.038	0.125	0.024	0.135	0.028	0.145	0.031
20	5	0.7	γ_1	0.132	0.032	0.151	0.036	0.129	0.029	0.164	0.052	0.161	0.043	0.125	0.024	0.180	0.051
			γ_2	0.321	0.181	0.373	0.221	0.312	0.167	0.390	0.282	0.386	0.238	0.305	0.141	0.439	0.293
			γ_3	0.417	0.304	0.484	0.372	0.405	0.281	0.507	0.475	0.501	0.401	0.396	0.238	0.570	0.494
			γ_4	0.470	0.382	0.550	0.481	0.456	0.355	0.566	0.584	0.561	0.499	0.447	0.302	0.643	0.625
40	10		γ_1	0.081	0.011	0.093	0.014	0.088	0.016	0.110	0.019	0.100	0.015	0.092	0.014	0.101	0.016
			γ_2	0.196	0.065	0.230	0.086	0.213	0.090	0.268	0.109	0.244	0.088	0.224	0.080	0.248	0.097
			γ_3	0.254	0.110	0.299	0.146	0.277	0.152	0.348	0.183	0.316	0.148	0.291	0.135	0.321	0.164
			γ_4	0.287	0.139	0.339	0.189	0.311	0.190	0.392	0.232	0.357	0.188	0.329	0.171	0.363	0.209
80	20		γ_1	0.062	0.006	0.063	0.007	0.066	0.007	0.059	0.006	0.070	0.008	0.067	0.007	0.076	0.009
			γ_2	0.151	0.036	0.154	0.041	0.161	0.041	0.145	0.036	0.172	0.045	0.164	0.040	0.185	0.053
			γ_3	0.196	0.060	0.201	0.069	0.209	0.070	0.188	0.060	0.224	0.076	0.213	0.068	0.240	0.089
			γ_4	0.222	0.077	0.228	0.090	0.236	0.089	0.212	0.077	0.253	0.097	0.241	0.087	0.271	0.113
120	30		γ_1	0.052	0.004	0.052	0.004	0.052	0.005	0.052	0.004	0.057	0.005	0.048	0.004	0.054	0.004
			γ_2	0.127	0.025	0.128	0.026	0.126	0.029	0.126	0.024	0.139	0.031	0.117	0.022	0.133	0.026
			γ_3	0.165	0.042	0.167	0.044	0.164	0.048	0.163	0.041	0.180	0.052	0.152	0.037	0.172	0.044
			γ_4	0.187	0.053	0.189	0.057	0.185	0.062	0.184	0.052	0.203	0.067	0.172	0.047	0.195	0.056
200	50		γ_1	0.032	0.002	0.038	0.002	0.043	0.003	0.038	0.003	0.044	0.003	0.043	0.003	0.043	0.003
			γ_2	0.079	0.010	0.095	0.013	0.104	0.017	0.094	0.016	0.107	0.019	0.105	0.018	0.105	0.018
			γ_3	0.102	0.017	0.123	0.021	0.135	0.029	0.122	0.027	0.139	0.032	0.136	0.030	0.136	0.030
			γ_4	0.116	0.021	0.139	0.027	0.153	0.037	0.138	0.035	0.157	0.041	0.154	0.039	0.154	0.039
20	5	0	γ_1	0.156	0.045	0.184	0.054	0.133	0.034	0.190	0.056	0.166	0.043	0.136	0.031	0.158	0.044
		(SRS)	γ_2	0.374	0.242	0.453	0.315	0.318	0.184	0.452	0.310	0.401	0.245	0.331	0.181	0.381	0.250
			γ_3	0.486	0.407	0.588	0.531	0.413	0.310	0.587	0.522	0.520	0.413	0.429	0.306	0.495	0.420
			γ_4	0.545	0.500	0.667	0.678	0.462	0.383	0.656	0.647	0.585	0.519	0.485	0.388	0.556	0.525
40	10		γ_1	0.105	0.018	0.112	0.021	0.111	0.023	0.110	0.019	0.104	0.018	0.100	0.016	0.118	0.023
			γ_2	0.256	0.104	0.273	0.122	0.267	0.129	0.267	0.110	0.251	0.104	0.244	0.096	0.287	0.136
			γ_3	0.332	0.176	0.354	0.206	0.346	0.218	0.346	0.186	0.326	0.176	0.316	0.162	0.373	0.229
			γ_4	0.375	0.223	0.400	0.262	0.389	0.273	0.389	0.234	0.368	0.222	0.356	0.205	0.420	0.289
80	20		γ_1	0.071	0.008	0.060	0.007	0.067	0.008	0.075	0.009	0.068	0.007	0.074	0.009	0.068	0.008
			γ_2	0.175	0.050	0.147	0.041	0.164	0.045	0.182	0.055	0.166	0.041	0.180	0.051	0.166	0.045
			γ_3	0.227	0.085	0.191	0.069	0.213	0.076	0.237	0.093	0.216	0.068	0.233	0.086	0.215	0.076
			γ_4	0.257	0.109	0.216	0.088	0.241	0.096	0.268	0.119	0.244	0.087	0.264	0.109	0.244	0.097
120	30		γ_1	0.052	0.004	0.060	0.007	0.058	0.006	0.068	0.007	0.053	0.005	0.058	0.005	0.065	0.007
			γ_2	0.128	0.025	0.146	0.041	0.143	0.033	0.167	0.042	0.129	0.028	0.142	0.028	0.159	0.039
			γ_3	0.166	0.043	0.190	0.069	0.185	0.056	0.216	0.071	0.167	0.047	0.185	0.047	0.206	0.065
			γ_4	0.187	0.055	0.215	0.087	0.209	0.072	0.245	0.091	0.189	0.060	0.209	0.060	0.233	0.082
200	50		γ_1	0.031	0.002	0.042	0.003	0.043	0.003	0.042	0.003	0.040	0.002	0.045	0.003	0.051	0.005
			γ_2	0.077	0.009	0.103	0.015	0.106	0.019	0.103	0.017	0.097	0.015	0.110	0.020	0.125	0.027
			γ_3	0.100	0.016	0.133	0.026	0.137	0.031	0.133	0.028	0.126	0.025	0.143	0.033	0.162	0.046
			γ_4	0.113	0.021	0.151	0.033	0.155	0.040	0.151	0.036	0.143	0.032	0.161	0.042	0.183	0.058

Table A.6
ABs and MSEs of the different entropy measures for IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 2 at $t = 4$

Set 2																	
t^*	d	ρ	Entropy-Type	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	1	γ_1	0.274	0.109	0.308	0.138	0.151	0.038	0.304	0.140	0.302	0.137	0.268	0.109	0.323	0.152
			γ_2	0.321	0.150	0.345	0.168	0.183	0.058	0.362	0.195	0.371	0.213	0.322	0.160	0.380	0.207
			γ_3	0.417	0.252	0.448	0.283	0.237	0.098	0.470	0.329	0.481	0.359	0.418	0.269	0.494	0.348
			γ_4	0.344	0.173	0.363	0.184	0.198	0.070	0.391	0.228	0.405	0.260	0.349	0.190	0.409	0.238
40	10		γ_1	0.192	0.054	0.204	0.066	0.104	0.017	0.209	0.066	0.190	0.057	0.189	0.052	0.204	0.062
			γ_2	0.229	0.078	0.234	0.085	0.121	0.023	0.252	0.097	0.230	0.087	0.227	0.077	0.243	0.089
			γ_3	0.297	0.132	0.303	0.142	0.157	0.039	0.328	0.164	0.298	0.147	0.295	0.130	0.315	0.150

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Table A.6 (*continued*)

Set 2																	
			γ_4	0.247	0.092	0.248	0.094	0.130	0.026	0.274	0.116	0.250	0.105	0.247	0.092	0.262	0.105
80	20		γ_1	0.141	0.032	0.151	0.035	0.092	0.014	0.144	0.033	0.118	0.023	0.142	0.030	0.143	0.034
			γ_2	0.166	0.044	0.175	0.046	0.109	0.020	0.171	0.047	0.138	0.032	0.169	0.043	0.169	0.048
			γ_3	0.216	0.074	0.227	0.078	0.142	0.033	0.223	0.080	0.180	0.053	0.220	0.072	0.219	0.080
			γ_4	0.178	0.050	0.186	0.052	0.118	0.023	0.185	0.055	0.148	0.036	0.183	0.050	0.181	0.055
120	30		γ_1	0.110	0.019	0.101	0.017	0.066	0.007	0.118	0.022	0.099	0.017	0.104	0.016	0.115	0.021
			γ_2	0.129	0.026	0.118	0.023	0.078	0.010	0.141	0.031	0.117	0.024	0.123	0.022	0.137	0.030
			γ_3	0.168	0.044	0.153	0.039	0.101	0.016	0.183	0.052	0.152	0.040	0.160	0.038	0.178	0.050
			γ_4	0.138	0.030	0.126	0.026	0.084	0.011	0.151	0.036	0.126	0.028	0.133	0.026	0.148	0.034
200	50		γ_1	0.093	0.013	0.090	0.011	0.049	0.004	0.098	0.015	0.084	0.011	0.082	0.010	0.097	0.013
			γ_2	0.109	0.018	0.106	0.016	0.058	0.005	0.117	0.022	0.100	0.016	0.097	0.015	0.115	0.018
			γ_3	0.142	0.031	0.137	0.027	0.076	0.009	0.151	0.037	0.130	0.027	0.126	0.025	0.149	0.031
			γ_4	0.117	0.021	0.114	0.018	0.063	0.006	0.126	0.026	0.108	0.019	0.105	0.017	0.123	0.021
20	5	0.9	γ_1	0.273	0.114	0.297	0.153	0.176	0.050	0.300	0.134	0.317	0.154	0.275	0.121	0.295	0.138
			γ_2	0.320	0.156	0.329	0.180	0.207	0.070	0.358	0.187	0.389	0.240	0.330	0.179	0.348	0.192
			γ_3	0.415	0.262	0.428	0.303	0.269	0.117	0.464	0.316	0.505	0.404	0.428	0.301	0.451	0.324
			γ_4	0.343	0.180	0.345	0.194	0.222	0.081	0.386	0.219	0.426	0.294	0.357	0.213	0.374	0.225
40	10		γ_1	0.198	0.061	0.235	0.082	0.127	0.026	0.260	0.104	0.222	0.070	0.202	0.060	0.203	0.067
			γ_2	0.232	0.083	0.268	0.104	0.151	0.037	0.312	0.153	0.263	0.099	0.242	0.087	0.241	0.095
			γ_3	0.301	0.140	0.348	0.176	0.196	0.063	0.405	0.258	0.342	0.167	0.314	0.147	0.313	0.160
			γ_4	0.249	0.096	0.284	0.116	0.163	0.044	0.339	0.182	0.284	0.115	0.262	0.103	0.260	0.111
80	20		γ_1	0.128	0.024	0.155	0.035	0.079	0.011	0.175	0.046	0.137	0.029	0.144	0.029	0.154	0.037
			γ_2	0.152	0.034	0.179	0.046	0.094	0.015	0.209	0.066	0.162	0.041	0.172	0.042	0.184	0.053
			γ_3	0.197	0.058	0.232	0.078	0.122	0.026	0.271	0.112	0.211	0.069	0.223	0.071	0.239	0.089
			γ_4	0.163	0.040	0.190	0.052	0.101	0.018	0.226	0.078	0.175	0.048	0.185	0.049	0.199	0.062
120	30		γ_1	0.114	0.019	0.126	0.025	0.071	0.008	0.130	0.027	0.116	0.021	0.100	0.016	0.110	0.020
			γ_2	0.135	0.026	0.146	0.033	0.084	0.011	0.156	0.038	0.137	0.030	0.118	0.022	0.129	0.027
			γ_3	0.175	0.044	0.189	0.056	0.109	0.019	0.202	0.065	0.177	0.051	0.154	0.038	0.167	0.045
			γ_4	0.145	0.030	0.155	0.037	0.090	0.013	0.168	0.045	0.147	0.035	0.127	0.026	0.138	0.031
200	50		γ_1	0.099	0.014	0.082	0.011	0.062	0.006	0.101	0.014	0.085	0.011	0.089	0.014	0.092	0.015
			γ_2	0.117	0.019	0.096	0.015	0.074	0.009	0.120	0.020	0.101	0.016	0.106	0.019	0.109	0.021
			γ_3	0.152	0.033	0.125	0.026	0.096	0.015	0.155	0.034	0.131	0.027	0.137	0.033	0.142	0.036
			γ_4	0.127	0.023	0.103	0.017	0.079	0.010	0.129	0.023	0.109	0.019	0.114	0.023	0.117	0.024
20	5	0.7	γ_1	0.305	0.154	0.342	0.172	0.198	0.067	0.373	0.206	0.321	0.162	0.266	0.121	0.312	0.156
			γ_2	0.359	0.219	0.384	0.210	0.238	0.101	0.447	0.303	0.389	0.243	0.322	0.179	0.365	0.211
			γ_3	0.466	0.370	0.498	0.354	0.308	0.170	0.580	0.510	0.505	0.409	0.418	0.302	0.473	0.355
			γ_4	0.387	0.259	0.404	0.231	0.257	0.122	0.485	0.363	0.423	0.293	0.350	0.215	0.391	0.242
40	10		γ_1	0.206	0.066	0.221	0.076	0.153	0.043	0.254	0.094	0.246	0.094	0.216	0.067	0.241	0.095
			γ_2	0.241	0.091	0.254	0.098	0.181	0.062	0.305	0.137	0.294	0.136	0.257	0.094	0.282	0.128
			γ_3	0.312	0.153	0.330	0.165	0.235	0.104	0.395	0.231	0.381	0.230	0.333	0.159	0.366	0.215
			γ_4	0.258	0.105	0.270	0.110	0.196	0.072	0.330	0.162	0.318	0.161	0.277	0.110	0.302	0.147
80	20		γ_1	0.159	0.035	0.170	0.049	0.111	0.021	0.154	0.040	0.163	0.043	0.135	0.030	0.165	0.042
			γ_2	0.187	0.048	0.195	0.062	0.133	0.030	0.182	0.057	0.194	0.061	0.160	0.043	0.195	0.059
			γ_3	0.243	0.081	0.253	0.104	0.172	0.050	0.236	0.096	0.252	0.102	0.208	0.072	0.253	0.099
			γ_4	0.201	0.056	0.206	0.069	0.143	0.035	0.196	0.066	0.209	0.071	0.173	0.051	0.210	0.068
120	30		γ_1	0.118	0.022	0.140	0.029	0.074	0.011	0.149	0.033	0.134	0.031	0.125	0.024	0.148	0.035
			γ_2	0.140	0.031	0.164	0.040	0.089	0.016	0.179	0.048	0.158	0.043	0.150	0.035	0.175	0.049
			γ_3	0.181	0.052	0.212	0.067	0.115	0.026	0.232	0.081	0.205	0.072	0.194	0.059	0.228	0.083
			γ_4	0.150	0.035	0.175	0.045	0.096	0.018	0.194	0.057	0.170	0.049	0.162	0.041	0.189	0.057
200	50		γ_1	0.084	0.011	0.088	0.012	0.065	0.008	0.106	0.019	0.103	0.017	0.098	0.016	0.099	0.015
			γ_2	0.099	0.015	0.104	0.016	0.077	0.011	0.126	0.026	0.123	0.024	0.117	0.023	0.117	0.021
			γ_3	0.129	0.025	0.135	0.027	0.100	0.018	0.163	0.044	0.159	0.040	0.152	0.038	0.152	0.036
			γ_4	0.107	0.017	0.111	0.019	0.083	0.012	0.135	0.031	0.133	0.028	0.126	0.026	0.126	0.025
20	5	0	γ_1	0.321	0.165	0.350	0.199	0.229	0.080	0.373	0.208	0.354	0.179	0.340	0.183	0.352	0.199
		(SRS)	γ_2	0.373	0.221	0.388	0.234	0.273	0.113	0.445	0.305	0.422	0.255	0.413	0.281	0.407	0.265
			γ_3	0.484	0.372	0.503	0.393	0.354	0.191	0.578	0.513	0.548	0.429	0.535	0.473	0.529	0.446
			γ_4	0.399	0.264	0.406	0.252	0.294	0.132	0.482	0.365	0.457	0.301	0.449	0.342	0.435	0.305
40	10		γ_1	0.255	0.104	0.259	0.103	0.161	0.040	0.223	0.081	0.265	0.104	0.271	0.109	0.261	0.107
			γ_2	0.298	0.140	0.296	0.132	0.190	0.055	0.267	0.118	0.320	0.156	0.328	0.165	0.304	0.144
			γ_3	0.386	0.235	0.384	0.222	0.246	0.093	0.347	0.199	0.416	0.262	0.425	0.278	0.395	0.242
			γ_4	0.318	0.160	0.314	0.147	0.204	0.064	0.289	0.140	0.348	0.187	0.356	0.199	0.326	0.165
80	20		γ_1	0.156	0.037	0.166	0.046	0.118	0.023	0.200	0.060	0.159	0.042	0.160	0.041	0.201	0.066
			γ_2	0.184	0.050	0.192	0.059	0.140	0.033	0.238	0.086	0.190	0.059	0.189	0.058	0.238	0.092
			γ_3	0.238	0.084	0.249	0.100	0.182	0.056	0.309	0.145	0.247	0.100	0.246	0.098	0.309	0.154
			γ_4	0.197	0.057	0.204	0.066	0.151	0.039	0.257	0.101	0.205	0.069	0.204	0.068	0.256	0.107
120	30		γ_1	0.144	0.033	0.147	0.035	0.103	0.017	0.168	0.043	0.142	0.033	0.151	0.034	0.144	0.033
			γ_2	0.171	0.046	0.170	0.046	0.122	0.024	0.198	0.062	0.168	0.046	0.180	0.049	0.170	0.046
			γ_3	0.222	0.078	0.221	0.078	0.158	0.040	0.257	0.104	0.218	0.077	0.234	0.082	0.220	0.077
			γ_4	0.184	0.054	0.182	0.052	0.131	0.028	0.213	0.072	0.181	0.053	0.195	0.057	0.183	0.053
200	50		γ_1	0.106	0.017	0.118	0.021	0.078	0.011	0.123	0.023	0.097	0.014	0.111	0.018	0.130	0.026
			γ_2	0.126	0.025	0.139	0.028	0.092	0.016	0.146	0.033	0.115	0.020	0.131	0.026	0.154	0.036
			γ_3	0.164	0.041	0.180	0.048	0.120	0.026	0.190	0.056	0.149	0.034	0.170	0.043	0.200	0.061
			γ_4	0.136	0.029	0.149	0.033	0.099	0.018	0.158	0.039	0.124	0.023	0.141	0.030	0.166	0.042

Table A.7
ABs and MSEs of the different entropy measures for IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 3 at $t = 4$

Set 3																	
t^*	d	ρ	Entropy-Type	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	1	γ_1	0.136	0.030	0.151	0.037	0.149	0.034	0.140	0.033	0.141	0.029	0.151	0.035	0.156	0.043
			γ_2	0.031	0.002	0.036	0.002	0.031	0.001	0.032	0.002	0.031	0.001	0.034	0.002	0.036	0.003
			γ_3	0.062	0.007	0.071	0.009	0.062	0.006	0.064	0.008	0.061	0.006	0.068	0.008	0.072	0.011
			γ_4	0.065	0.007	0.073	0.009	0.068	0.007	0.067	0.008	0.066	0.007	0.072	0.008	0.074	0.011
40	10		γ_1	0.107	0.018	0.106	0.018	0.093	0.014	0.087	0.013	0.099	0.014	0.084	0.012	0.105	0.015
			γ_2	0.023	0.001	0.024	0.001	0.021	0.001	0.018	0.001	0.021	0.001	0.018	0.001	0.023	0.001
			γ_3	0.046	0.003	0.047	0.004	0.042	0.003	0.037	0.002	0.042	0.002	0.036	0.002	0.046	0.003
			γ_4	0.050	0.004	0.050	0.004	0.044	0.003	0.040	0.003	0.046	0.003	0.039	0.003	0.049	0.003
80	20		γ_1	0.073	0.009	0.074	0.009	0.082	0.010	0.069	0.008	0.070	0.008	0.075	0.009	0.079	0.010
			γ_2	0.016	0.000	0.016	0.000	0.018	0.000	0.015	0.000	0.015	0.000	0.016	0.000	0.017	0.001
			γ_3	0.032	0.002	0.033	0.002	0.035	0.002	0.029	0.001	0.030	0.001	0.032	0.002	0.035	0.002
			γ_4	0.034	0.002	0.035	0.002	0.038	0.002	0.032	0.002	0.032	0.002	0.035	0.002	0.037	0.002
120	30		γ_1	0.063	0.006	0.048	0.004	0.065	0.006	0.057	0.005	0.054	0.005	0.058	0.005	0.057	0.005
			γ_2	0.014	0.000	0.011	0.000	0.014	0.000	0.012	0.000	0.012	0.000	0.013	0.000	0.012	0.000
			γ_3	0.027	0.001	0.021	0.001	0.028	0.001	0.025	0.001	0.023	0.001	0.025	0.001	0.024	0.001
			γ_4	0.029	0.001	0.023	0.001	0.030	0.001	0.026	0.001	0.025	0.001	0.027	0.001	0.026	0.001
200	50		γ_1	0.046	0.003	0.045	0.003	0.046	0.003	0.046	0.003	0.040	0.002	0.044	0.003	0.048	0.003
			γ_2	0.010	0.000	0.010	0.000	0.010	0.000	0.010	0.000	0.008	0.000	0.010	0.000	0.010	0.000
			γ_3	0.020	0.001	0.019	0.001	0.020	0.001	0.020	0.001	0.017	0.000	0.019	0.001	0.020	0.001
			γ_4	0.022	0.001	0.021	0.001	0.021	0.001	0.021	0.001	0.018	0.001	0.021	0.001	0.022	0.001
20	5	0.9	γ_1	0.156	0.037	0.167	0.046	0.162	0.040	0.171	0.046	0.157	0.041	0.152	0.039	0.159	0.043
			γ_2	0.034	0.002	0.040	0.003	0.037	0.002	0.039	0.003	0.035	0.002	0.034	0.002	0.035	0.002
			γ_3	0.069	0.008	0.080	0.012	0.074	0.009	0.078	0.012	0.069	0.009	0.068	0.008	0.071	0.010
			γ_4	0.073	0.008	0.081	0.012	0.077	0.009	0.082	0.011	0.073	0.009	0.071	0.009	0.075	0.010
40	10		γ_1	0.118	0.020	0.116	0.022	0.123	0.022	0.122	0.024	0.118	0.021	0.115	0.020	0.111	0.019
			γ_2	0.026	0.001	0.026	0.001	0.027	0.001	0.027	0.001	0.026	0.001	0.025	0.001	0.024	0.001
			γ_3	0.052	0.004	0.053	0.005	0.053	0.004	0.053	0.005	0.052	0.004	0.050	0.004	0.048	0.004
			γ_4	0.055	0.004	0.055	0.005	0.057	0.005	0.057	0.005	0.056	0.005	0.053	0.004	0.052	0.004
80	20		γ_1	0.066	0.007	0.085	0.011	0.076	0.009	0.100	0.014	0.075	0.009	0.079	0.010	0.079	0.010
			γ_2	0.014	0.000	0.019	0.001	0.016	0.000	0.022	0.001	0.016	0.000	0.017	0.000	0.017	0.001
			γ_3	0.029	0.001	0.038	0.002	0.033	0.002	0.043	0.003	0.033	0.002	0.034	0.002	0.034	0.002
			γ_4	0.031	0.001	0.040	0.003	0.035	0.002	0.047	0.003	0.035	0.002	0.037	0.002	0.037	0.002
120	30		γ_1	0.062	0.006	0.066	0.007	0.070	0.007	0.073	0.009	0.068	0.008	0.057	0.005	0.066	0.007
			γ_2	0.014	0.000	0.015	0.000	0.015	0.000	0.016	0.000	0.015	0.000	0.012	0.000	0.014	0.000
			γ_3	0.027	0.001	0.029	0.002	0.031	0.001	0.031	0.002	0.030	0.002	0.025	0.001	0.029	0.001
			γ_4	0.029	0.001	0.031	0.002	0.033	0.002	0.034	0.002	0.032	0.002	0.027	0.001	0.031	0.001
200	50		γ_1	0.057	0.005	0.053	0.004	0.059	0.005	0.047	0.003	0.050	0.004	0.052	0.004	0.050	0.004
			γ_2	0.012	0.000	0.011	0.000	0.013	0.000	0.010	0.000	0.011	0.000	0.011	0.000	0.011	0.000
			γ_3	0.024	0.001	0.023	0.001	0.025	0.001	0.020	0.001	0.022	0.001	0.022	0.001	0.022	0.001
			γ_4	0.026	0.001	0.024	0.001	0.027	0.001	0.022	0.001	0.023	0.001	0.024	0.001	0.024	0.001
20	5	0.7	γ_1	0.187	0.059	0.205	0.061	0.183	0.052	0.192	0.063	0.193	0.056	0.164	0.041	0.197	0.063
			γ_2	0.042	0.003	0.047	0.003	0.040	0.003	0.045	0.004	0.042	0.003	0.036	0.002	0.045	0.004
			γ_3	0.085	0.013	0.094	0.013	0.080	0.010	0.090	0.017	0.085	0.011	0.072	0.008	0.090	0.015
			γ_4	0.089	0.014	0.098	0.014	0.085	0.011	0.092	0.016	0.090	0.012	0.077	0.009	0.094	0.015
40	10		γ_1	0.128	0.025	0.135	0.030	0.143	0.037	0.128	0.025	0.122	0.024	0.124	0.024	0.128	0.028
			γ_2	0.029	0.001	0.030	0.002	0.032	0.002	0.028	0.001	0.027	0.001	0.027	0.001	0.029	0.002
			γ_3	0.057	0.005	0.060	0.006	0.063	0.008	0.057	0.005	0.054	0.005	0.055	0.005	0.058	0.006
			γ_4	0.060	0.006	0.063	0.007	0.067	0.008	0.060	0.006	0.057	0.005	0.058	0.005	0.061	0.007
80	20		γ_1	0.093	0.013	0.089	0.013	0.106	0.018	0.079	0.011	0.086	0.011	0.089	0.012	0.096	0.014
			γ_2	0.020	0.001	0.020	0.001	0.022	0.001	0.017	0.001	0.019	0.001	0.019	0.001	0.021	0.001
			γ_3	0.041	0.003	0.040	0.003	0.045	0.003	0.034	0.002	0.038	0.002	0.038	0.002	0.042	0.003
			γ_4	0.044	0.003	0.042	0.003	0.049	0.004	0.037	0.002	0.040	0.003	0.041	0.003	0.045	0.003
120	30		γ_1	0.072	0.009	0.083	0.011	0.072	0.008	0.081	0.010	0.079	0.011	0.066	0.006	0.082	0.011
			γ_2	0.015	0.000	0.018	0.001	0.015	0.000	0.017	0.000	0.017	0.001	0.014	0.000	0.018	0.001
			γ_3	0.031	0.002	0.036	0.002	0.031	0.002	0.035	0.002	0.035	0.002	0.029	0.001	0.035	0.002
			γ_4	0.033	0.002	0.039	0.002	0.033	0.002	0.037	0.002	0.037	0.002	0.031	0.001	0.038	0.002
200	50		γ_1	0.055	0.004	0.056	0.004	0.066	0.007	0.053	0.005	0.059	0.005	0.059	0.005	0.056	0.005
			γ_2	0.012	0.000	0.012	0.000	0.014	0.000	0.011	0.000	0.013	0.000	0.013	0.000	0.012	0.000
			γ_3	0.024	0.001	0.024	0.001	0.029	0.001	0.023	0.001	0.025	0.001	0.026	0.001	0.024	0.001
			γ_4	0.025	0.001	0.026	0.001	0.031	0.002	0.024	0.001	0.027	0.001	0.028	0.001	0.026	0.001
20	5	0	γ_1	0.206	0.069	0.210	0.075	0.220	0.074	0.244	0.088	0.211	0.070	0.203	0.071	0.220	0.078
		(SRS)	γ_2	0.048	0.004	0.049	0.004	0.048	0.004	0.057	0.005	0.047	0.004	0.043	0.003	0.051	0.005
			γ_3	0.096	0.017	0.097	0.018	0.097	0.016	0.115	0.021	0.094	0.015	0.086	0.013	0.102	0.018
			γ_4	0.099	0.017	0.100	0.018	0.103	0.016	0.118	0.021	0.099	0.016	0.093	0.015	0.105	0.018
40	10		γ_1	0.153	0.039	0.164	0.043	0.161	0.040	0.130	0.026	0.167	0.042	0.150	0.035	0.159	0.040
			γ_2	0.035	0.002	0.037	0.002	0.037	0.002	0.029	0.001	0.036	0.002	0.033	0.002	0.036	0.002
			γ_3	0.069	0.009	0.074	0.009	0.074	0.009	0.058	0.006	0.072	0.008	0.065	0.007	0.072	0.009
			γ_4	0.072	0.009	0.078	0.010	0.077	0.009	0.061	0.006	0.077	0.009	0.070	0.008	0.076	0.009
80	20		γ_1	0.105	0.017	0.098	0.017	0.117	0.020	0.106	0.017	0.096	0.015	0.112	0.019	0.113	0.023
			γ_2	0.023	0.001	0.022	0.001	0.026	0.001	0.023	0.001	0.021	0.001	0.024	0.001	0.024	0.001
			γ_3	0.046	0.003	0.044	0.004	0.052	0.004	0.045	0.003	0.042	0.003	0.049	0.004	0.049	0.004
			γ_4	0.049	0.004	0.046	0.004	0.055	0.005	0.049	0.004	0.045	0.003	0.052	0.004	0.052	0.005

(*continued on next page*)

Table A.7 (*continued*)

Set 3																	
120	30		γ_1	0.093	0.014	0.093	0.015	0.097	0.014	0.105	0.017	0.089	0.013	0.093	0.014	0.093	0.014
			γ_2	0.020	0.001	0.021	0.001	0.021	0.001	0.023	0.001	0.019	0.001	0.020	0.001	0.021	0.001
			γ_3	0.040	0.003	0.041	0.003	0.042	0.003	0.046	0.003	0.038	0.002	0.039	0.002	0.041	0.003
			γ_4	0.043	0.003	0.044	0.003	0.045	0.003	0.049	0.004	0.041	0.003	0.043	0.003	0.044	0.003
200	50		γ_1	0.059	0.006	0.070	0.007	0.079	0.010	0.070	0.007	0.058	0.005	0.070	0.008	0.073	0.008
			γ_2	0.013	0.000	0.015	0.000	0.017	0.000	0.015	0.000	0.012	0.000	0.015	0.000	0.016	0.000
			γ_3	0.025	0.001	0.030	0.001	0.034	0.002	0.030	0.001	0.025	0.001	0.030	0.001	0.032	0.002
			γ_4	0.027	0.001	0.032	0.002	0.037	0.002	0.032	0.002	0.027	0.001	0.033	0.002	0.034	0.002

Table A.8
ABs and MSEs of the different entropy measures for IEP distribution under perfect and imperfect RSS and SRS using different methods of estimations for Set 4 at $t = 4$

Set 4																	
t^*	d	ρ	Entropy-Type	ML		MPS		KM		LS		WLS		AD		CVM	
				AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	5	1	γ_1	0.123	0.024	0.141	0.032	0.111	0.019	0.124	0.024	0.127	0.023	0.135	0.029	0.139	0.033
			γ_2	0.102	0.018	0.122	0.026	0.085	0.010	0.102	0.018	0.102	0.016	0.112	0.022	0.117	0.027
			γ_3	0.204	0.070	0.244	0.104	0.170	0.042	0.204	0.071	0.203	0.064	0.224	0.087	0.233	0.107
			γ_4	0.112	0.020	0.131	0.029	0.097	0.014	0.112	0.021	0.113	0.019	0.123	0.025	0.127	0.029
40	10		γ_1	0.091	0.013	0.092	0.014	0.070	0.008	0.079	0.010	0.086	0.011	0.075	0.009	0.092	0.012
			γ_2	0.072	0.008	0.076	0.010	0.057	0.006	0.062	0.006	0.067	0.006	0.058	0.006	0.074	0.008
			γ_3	0.144	0.032	0.152	0.042	0.114	0.022	0.124	0.026	0.135	0.025	0.117	0.023	0.148	0.032
			γ_4	0.081	0.010	0.083	0.012	0.063	0.007	0.070	0.008	0.076	0.008	0.066	0.007	0.083	0.010
80	20		γ_1	0.065	0.007	0.066	0.007	0.059	0.005	0.061	0.006	0.059	0.006	0.066	0.007	0.069	0.008
			γ_2	0.052	0.004	0.054	0.005	0.046	0.003	0.048	0.004	0.047	0.004	0.052	0.004	0.056	0.005
			γ_3	0.103	0.018	0.108	0.019	0.092	0.013	0.096	0.015	0.095	0.015	0.104	0.016	0.111	0.021
			γ_4	0.058	0.005	0.060	0.006	0.052	0.004	0.054	0.005	0.053	0.005	0.059	0.005	0.062	0.006
120	30		γ_1	0.055	0.004	0.043	0.003	0.045	0.003	0.050	0.004	0.045	0.004	0.050	0.004	0.050	0.004
			γ_2	0.043	0.003	0.034	0.002	0.035	0.002	0.039	0.002	0.036	0.002	0.040	0.002	0.039	0.002
			γ_3	0.087	0.011	0.068	0.008	0.071	0.008	0.079	0.010	0.072	0.009	0.079	0.010	0.078	0.009
			γ_4	0.049	0.004	0.038	0.002	0.040	0.002	0.044	0.003	0.040	0.003	0.045	0.003	0.044	0.003
200	50		γ_1	0.040	0.003	0.038	0.002	0.036	0.002	0.040	0.002	0.034	0.002	0.038	0.002	0.043	0.003
			γ_2	0.032	0.002	0.030	0.001	0.028	0.001	0.031	0.002	0.027	0.001	0.030	0.001	0.034	0.002
			γ_3	0.065	0.007	0.061	0.006	0.057	0.005	0.062	0.006	0.054	0.005	0.060	0.006	0.067	0.007
			γ_4	0.036	0.002	0.034	0.002	0.032	0.002	0.035	0.002	0.030	0.001	0.034	0.002	0.038	0.002
20	5	0.9	γ_1	0.136	0.030	0.151	0.037	0.116	0.022	0.141	0.028	0.136	0.031	0.135	0.031	0.141	0.033
			γ_2	0.113	0.022	0.131	0.031	0.093	0.015	0.115	0.020	0.109	0.021	0.111	0.023	0.116	0.024
			γ_3	0.225	0.087	0.262	0.124	0.186	0.059	0.231	0.081	0.217	0.083	0.223	0.092	0.232	0.096
			γ_4	0.123	0.025	0.140	0.034	0.104	0.018	0.127	0.024	0.121	0.025	0.122	0.026	0.127	0.028
40	10		γ_1	0.107	0.016	0.108	0.019	0.082	0.010	0.110	0.019	0.101	0.015	0.095	0.014	0.095	0.015
			γ_2	0.086	0.011	0.090	0.014	0.064	0.006	0.088	0.012	0.082	0.010	0.075	0.009	0.077	0.010
			γ_3	0.172	0.044	0.181	0.055	0.128	0.026	0.176	0.049	0.163	0.042	0.150	0.035	0.154	0.040
			γ_4	0.096	0.013	0.099	0.016	0.072	0.008	0.098	0.015	0.091	0.013	0.084	0.011	0.085	0.012
80	20		γ_1	0.059	0.006	0.075	0.008	0.055	0.005	0.080	0.010	0.066	0.007	0.065	0.006	0.066	0.008
			γ_2	0.047	0.004	0.061	0.006	0.043	0.003	0.063	0.006	0.052	0.005	0.052	0.004	0.053	0.005
			γ_3	0.094	0.014	0.123	0.023	0.086	0.012	0.127	0.025	0.104	0.018	0.103	0.016	0.105	0.021
			γ_4	0.053	0.004	0.068	0.007	0.049	0.004	0.071	0.008	0.058	0.006	0.058	0.005	0.059	0.006
120	30		γ_1	0.054	0.004	0.058	0.006	0.051	0.004	0.063	0.007	0.056	0.005	0.049	0.004	0.055	0.005
			γ_2	0.043	0.003	0.047	0.004	0.041	0.003	0.050	0.004	0.044	0.003	0.039	0.003	0.044	0.003
			γ_3	0.086	0.011	0.095	0.016	0.082	0.011	0.099	0.016	0.088	0.012	0.078	0.010	0.088	0.012
			γ_4	0.048	0.004	0.053	0.005	0.046	0.003	0.056	0.005	0.049	0.004	0.044	0.003	0.049	0.004
200	50		γ_1	0.049	0.004	0.044	0.003	0.042	0.003	0.038	0.002	0.045	0.003	0.044	0.003	0.039	0.003
			γ_2	0.039	0.002	0.035	0.002	0.033	0.002	0.030	0.001	0.035	0.002	0.035	0.002	0.031	0.002
			γ_3	0.077	0.009	0.069	0.007	0.066	0.007	0.060	0.006	0.071	0.007	0.070	0.008	0.062	0.007
			γ_4	0.044	0.003	0.039	0.002	0.037	0.002	0.034	0.002	0.040	0.002	0.039	0.003	0.035	0.002
20	5	0.7	γ_1	0.154	0.043	0.178	0.047	0.140	0.030	0.159	0.044	0.168	0.041	0.134	0.030	0.164	0.044
			γ_2	0.127	0.032	0.153	0.038	0.113	0.021	0.134	0.036	0.137	0.030	0.107	0.021	0.140	0.037
			γ_3	0.254	0.127	0.306	0.151	0.227	0.083	0.268	0.145	0.275	0.118	0.215	0.083	0.279	0.147
			γ_4	0.139	0.036	0.164	0.042	0.125	0.025	0.145	0.039	0.152	0.034	0.120	0.025	0.151	0.040
40	10		γ_1	0.111	0.019	0.120	0.023	0.108	0.020	0.109	0.018	0.102	0.019	0.115	0.019	0.118	0.022
			γ_2	0.090	0.013	0.099	0.017	0.086	0.013	0.087	0.012	0.083	0.013	0.092	0.013	0.098	0.016
			γ_3	0.181	0.053	0.197	0.067	0.171	0.053	0.174	0.048	0.165	0.052	0.183	0.052	0.196	0.065
			γ_4	0.100	0.016	0.109	0.020	0.096	0.016	0.097	0.015	0.092	0.015	0.102	0.016	0.108	0.019
80	20		γ_1	0.080	0.010	0.080	0.011	0.084	0.011	0.072	0.008	0.076	0.008	0.072	0.008	0.081	0.011
			γ_2	0.065	0.006	0.066	0.008	0.066	0.007	0.058	0.005	0.061	0.006	0.057	0.005	0.066	0.007
			γ_3	0.129	0.025	0.132	0.031	0.132	0.027	0.115	0.022	0.122	0.022	0.114	0.021	0.132	0.028
			γ_4	0.072	0.008	0.073	0.009	0.075	0.009	0.064	0.007	0.068	0.007	0.064	0.006	0.073	0.009
120	30		γ_1	0.059	0.006	0.067	0.007	0.058	0.005	0.069	0.008	0.063	0.007	0.057	0.005	0.075	0.009
			γ_2	0.047	0.004	0.054	0.005	0.046	0.003	0.054	0.005	0.051	0.005	0.045	0.003	0.060	0.006
			γ_3	0.093	0.014	0.108	0.018	0.091	0.013	0.108	0.018	0.101	0.018	0.090	0.012	0.120	0.023
			γ_4	0.053	0.005	0.060	0.006	0.051	0.004	0.061	0.006	0.056	0.006	0.051	0.004	0.067	0.007
200	50		γ_1	0.045	0.003	0.046	0.003	0.052	0.004	0.047	0.003	0.050	0.004	0.049	0.004	0.048	0.004

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Table A.8 (*continued*)

Set 4																	
			γ_2	0.036	0.002	0.037	0.002	0.041	0.003	0.037	0.002	0.039	0.002	0.039	0.002	0.038	0.002
			γ_3	0.071	0.007	0.073	0.008	0.082	0.011	0.074	0.008	0.078	0.009	0.078	0.010	0.077	0.009
			γ_4	0.040	0.002	0.041	0.003	0.046	0.003	0.041	0.003	0.044	0.003	0.044	0.003	0.043	0.003
20	5	0	γ_1	0.180	0.054	0.178	0.055	0.168	0.044	0.216	0.070	0.182	0.051	0.175	0.052	0.190	0.058
		(SRS)	γ_2	0.154	0.045	0.156	0.047	0.137	0.033	0.187	0.057	0.151	0.039	0.139	0.033	0.161	0.045
			γ_3	0.308	0.179	0.312	0.189	0.273	0.131	0.374	0.230	0.302	0.156	0.278	0.134	0.323	0.180
			γ_4	0.166	0.048	0.166	0.050	0.151	0.037	0.200	0.062	0.165	0.044	0.155	0.041	0.174	0.050
40	10		γ_1	0.131	0.029	0.140	0.031	0.124	0.023	0.114	0.020	0.143	0.031	0.132	0.026	0.139	0.030
			γ_2	0.108	0.022	0.117	0.023	0.102	0.017	0.093	0.014	0.113	0.020	0.105	0.017	0.115	0.023
			γ_3	0.217	0.086	0.235	0.093	0.205	0.066	0.186	0.057	0.226	0.078	0.210	0.068	0.231	0.090
			γ_4	0.119	0.025	0.128	0.027	0.112	0.019	0.103	0.017	0.127	0.024	0.117	0.021	0.126	0.026
80	20		γ_1	0.087	0.012	0.084	0.012	0.087	0.011	0.092	0.013	0.082	0.011	0.093	0.013	0.098	0.017
			γ_2	0.070	0.008	0.069	0.009	0.070	0.008	0.073	0.008	0.065	0.007	0.074	0.009	0.078	0.011
			γ_3	0.140	0.032	0.139	0.036	0.139	0.031	0.146	0.033	0.129	0.028	0.149	0.034	0.156	0.044
			γ_4	0.078	0.010	0.076	0.011	0.078	0.009	0.082	0.010	0.073	0.009	0.083	0.011	0.088	0.014
120	30		γ_1	0.079	0.010	0.079	0.011	0.074	0.008	0.088	0.012	0.075	0.009	0.079	0.010	0.080	0.010
			γ_2	0.062	0.006	0.064	0.008	0.058	0.005	0.071	0.008	0.059	0.006	0.061	0.006	0.064	0.007
			γ_3	0.125	0.025	0.128	0.031	0.117	0.021	0.142	0.031	0.119	0.024	0.123	0.024	0.129	0.027
			γ_4	0.070	0.008	0.071	0.009	0.066	0.007	0.079	0.010	0.067	0.007	0.069	0.008	0.072	0.008
200	50		γ_1	0.052	0.004	0.060	0.005	0.058	0.005	0.060	0.005	0.049	0.004	0.059	0.006	0.063	0.006
			γ_2	0.040	0.003	0.047	0.003	0.046	0.003	0.047	0.003	0.039	0.002	0.047	0.004	0.050	0.004
			γ_3	0.080	0.010	0.095	0.013	0.092	0.014	0.095	0.014	0.077	0.009	0.094	0.014	0.101	0.016
			γ_4	0.045	0.003	0.053	0.004	0.052	0.004	0.053	0.004	0.043	0.003	0.053	0.004	0.057	0.005

Table A.9
RE of the parameter estimates for IEP distribution for Set1 and Set 2

Set 1										
t^*	d	ρ	Parameters	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	a	2.314	1.236	1.553	1.067	1.092	1.154	1.627
			b	1.771	1.557	1.949	1.456	1.203	1.204	1.456
40	10		a	3.084	1.196	1.330	1.474	2.047	1.105	1.937
			b	3.361	1.389	3.240	1.274	1.838	1.289	1.721
80	20		a	3.125	1.027	1.955	1.101	1.136	1.317	1.716
			b	1.850	1.850	1.669	1.669	2.754	1.762	1.820
120	30		a	1.158	1.275	1.794	1.268	1.297	1.371	1.769
			b	1.282	1.016	2.529	1.164	1.231	1.056	1.691
200	50		a	3.161	1.253	1.780	1.286	1.814	1.177	1.520
			b	1.092	1.182	1.347	1.107	1.160	1.059	1.243
20	5	0.9	a	1.571	1.220	1.232	1.008	1.787	1.604	3.811
			b	1.168	1.374	1.547	1.456	1.992	1.398	3.408
40	10		a	1.951	1.046	1.082	1.307	2.004	1.452	1.295
			b	1.096	1.064	1.678	1.005	1.242	1.408	1.208
80	20		a	1.299	1.079	2.077	1.123	1.281	1.182	1.555
			b	2.061	1.462	1.638	1.273	1.638	1.085	1.273
120	30		a	1.257	1.146	1.146	1.364	1.706	1.314	1.823
			b	1.098	1.382	1.382	1.117	1.173	1.020	1.552
200	50		a	1.446	1.495	2.253	1.174	1.629	1.083	1.467
			b	1.194	1.350	1.817	1.159	1.188	1.009	1.131
20	5	0.7	a	1.884	1.079	1.245	1.173	1.022	1.126	1.362
			b	1.313	1.216	1.374	1.064	1.156	1.652	1.334
40	10		a	2.089	1.170	1.342	1.184	1.263	1.627	1.320
			b	1.921	1.126	1.839	1.398	2.285	1.104	1.077
80	20		a	1.155	1.410	1.410	1.173	1.332	1.198	1.243
			b	1.700	1.113	1.113	1.909	1.921	1.246	1.568

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Table A.9 (*continued*)

t^*	d	ρ	Parameters	ML	MPS	KM	LS	WLS	AD	CVM
Set 1										
120	30		a	1.160	1.231	1.231	1.409	1.880	1.261	1.536
			b	1.243	1.555	1.555	1.086	1.004	1.088	1.810
200	50		a	1.483	1.258	1.687	1.250	1.613	1.085	1.355
			b	1.478	1.144	1.282	1.307	1.150	1.150	1.440
Set 2										
t^*	d	ρ	Parameters	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	a	1.546	1.703	1.204	1.956	3.910	1.184	1.082
			b	1.419	1.519	1.973	1.554	2.453	1.211	1.921
40	10		a	2.929	1.112	1.119	1.054	1.004	1.093	1.392
			b	2.469	1.213	1.544	1.458	1.864	1.113	1.321
80	20		a	1.012	1.327	1.887	1.295	1.391	1.307	1.818
			b	1.469	3.197	1.624	1.614	2.426	1.236	1.732
120	30		a	1.004	1.375	1.662	1.117	1.920	1.267	1.314
			b	1.311	1.329	2.768	1.466	1.620	1.234	1.421
200	50		a	1.873	1.211	1.613	1.091	1.141	1.092	1.256
			b	1.166	1.260	1.486	1.698	2.590	1.990	1.153
20	5	0.9	a	1.226	1.302	1.151	1.047	1.486	1.771	1.730
			b	1.067	1.309	1.660	1.921	1.609	1.551	2.365
40	10		a	1.267	0.911	1.106	1.983	1.706	1.392	1.245
			b	1.109	1.017	1.018	1.128	1.275	1.214	1.192
80	20		a	1.316	1.415	1.980	1.953	1.334	1.257	1.754
			b	1.569	1.570	1.858	1.232	1.858	1.054	1.232
120	30		a	0.982	1.063	1.063	1.170	1.151	1.239	1.299
			b	1.157	1.801	1.801	1.271	1.432	1.099	1.278
200	50		a	1.348	1.379	1.881	1.847	1.690	1.037	1.224
			b	1.363	1.310	1.637	1.573	2.149	1.967	1.046
20	5	0.7	a	1.134	1.349	1.147	1.032	1.076	1.048	1.093
			b	1.043	1.241	1.458	1.264	1.087	1.055	1.455
40	10		a	1.275	1.064	1.082	1.219	1.055	1.592	1.221
			b	1.210	1.025	1.022	1.078	1.153	1.111	1.216
80	20		a	1.186	1.187	1.187	1.971	1.436	1.232	1.404
			b	1.475	1.200	1.200	1.183	1.336	1.210	1.471
120	30		a	1.060	1.143	1.143	1.431	2.229	1.218	1.489
			b	1.247	1.712	1.712	1.259	1.464	1.219	1.504
200	50		a	1.223	1.266	1.628	1.157	1.300	1.067	1.164
			b	1.274	1.221	1.417	1.278	1.215	1.215	1.270

Table A.10
RE of the parameter estimates for IEP distribution for Set 3 and Set 4

t^*	d	ρ	Parameters	ML	MPS	KM	LS	WLS	AD	CVM
Set 3										
20	5	1	a	2.265	1.320	1.539	1.827	1.789	1.080	2.192
			b	1.432	1.530	1.979	1.989	1.006	1.204	1.210
40	10		a	3.001	1.131	1.155	1.372	1.734	1.082	1.378
			b	2.586	1.224	1.584	1.194	1.468	1.115	1.348
80	20		a	1.012	1.944	1.933	1.921	1.851	1.307	1.773
			b	1.467	2.875	2.825	1.992	1.858	1.240	1.742
120	30		a	1.068	1.119	1.288	1.964	1.841	1.275	1.377
			b	2.928	1.107	1.362	1.974	1.963	1.237	1.448

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Table A.10 (*continued*)

Set 3										
200	50		*a*	1.901	1.217	1.625	1.221	1.534	1.028	1.095
			b	2.889	1.250	1.448	1.107	1.311	2.879	1.799
20	5	0.9	*a*	1.482	1.197	1.200	1.773	2.454	1.098	1.064
			b	1.045	1.306	1.583	1.809	2.610	1.114	1.019
40	10		*a*	1.260	1.949	1.825	1.175	1.516	1.797	1.615
			b	1.002	1.053	1.019	1.939	1.901	1.016	1.034
80	20		*a*	1.249	1.007	1.982	1.064	1.242	1.208	1.592
			b	1.450	1.917	1.802	1.930	1.848	1.041	1.246
120	30		*a*	1.113	1.901	1.832	1.960	1.880	1.277	1.434
			b	1.839	1.924	1.948	1.940	1.827	1.155	1.365
200	50		*a*	1.367	1.380	1.862	1.043	1.088	1.983	1.059
			b	1.088	1.267	1.579	1.019	1.976	1.897	1.757
20	5	0.7	*a*	2.077	1.252	1.579	1.851	1.549	1.927	1.598
			b	1.259	1.293	1.521	1.899	1.652	1.992	1.787
40	10		*a*	1.438	1.037	1.056	1.156	1.472	1.930	1.020
			b	1.316	1.999	1.924	1.073	1.156	1.090	1.134
80	20		*a*	1.194	1.827	1.618	1.103	1.475	1.206	1.235
			b	1.585	1.796	1.621	1.061	1.074	1.179	1.425
120	30		*a*	1.188	1.894	1.841	1.094	1.154	1.169	1.393
			b	1.962	1.869	1.863	1.987	1.911	1.261	1.612
200	50		*a*	1.345	1.252	1.539	1.058	1.066	1.978	1.989
			b	1.051	1.253	1.461	1.906	1.873	1.915	1.835
Set 4										
t^*	d	ρ	Parameters	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	*a*	1.002	1.269	1.172	2.609	1.786	1.589	1.801
			b	1.333	1.512	1.960	1.395	1.950	1.231	1.454
40	10		*a*	1.118	1.630	1.235	1.114	2.277	1.550	1.424
			b	2.406	1.203	1.514	1.329	1.243	1.451	1.207
80	20		*a*	1.505	2.962	1.795	1.042	2.665	1.687	1.640
			b	1.464	1.879	1.846	1.727	1.174	1.304	1.209
120	30		*a*	1.322	3.717	1.753	5.767	3.898	2.306	1.800
			b	1.938	1.078	1.238	1.364	1.990	1.125	1.223
200	50		*a*	1.193	4.293	2.431	0.482	4.205	2.915	1.976
			b	1.917	1.264	1.500	1.824	1.060	1.091	1.208
20	5	0.9	*a*	1.801	1.089	1.099	2.142	2.872	2.665	1.754
			b	1.060	1.333	1.693	1.419	1.554	1.169	1.842
40	10		*a*	1.060	2.602	1.153	1.013	1.973	1.920	2.432
			b	1.013	1.145	1.185	1.506	2.200	1.509	1.164
80	20		*a*	1.097	2.557	1.531	1.184	9.522	1.477	1.926
			b	1.577	2.868	1.500	1.163	1.500	1.051	1.163
120	30		*a*	1.217	1.668	1.668	1.153	1.400	1.573	1.422
			b	1.833	2.897	1.897	1.392	2.000	1.268	1.579
200	50		*a*	1.746	5.017	3.430	3.326	5.600	1.423	1.508
			b	1.000	1.250	1.385	1.425	1.667	1.893	1.824
20	5	0.7	*a*	1.127	1.018	1.072	4.266	1.146	1.119	1.299
			b	1.294	1.382	1.626	1.931	1.049	1.984	1.364
40	10		*a*	1.026	1.017	1.049	4.392	1.099	1.037	1.008
			b	1.085	1.016	1.982	1.112	1.100	1.150	1.348
80	20		*a*	1.062	1.018	1.018	1.122	1.553	1.197	3.624
			b	1.708	1.660	1.660	1.013	1.000	1.192	1.541

(*continued on next page*)

Table A.10 (*continued*)

Set 3									
120	30	*a*	1.091	1.011	1.011	1.030	1.314	1.153	1.065
		b	1.938	1.813	1.813	1.076	1.000	1.193	1.500
200	50	*a*	1.899	1.115	1.047	1.041	1.120	1.098	1.066
		b	1.100	1.250	1.286	1.875	1.923	1.923	1.353

Table A.11
RE of the different entropy measures for IEP distribution for Set1

Set 1										
t^*	*d*	ρ	Entropy_Type	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	γ_1	2.644	1.674	2.733	1.606	3.050	1.589	2.366
			γ_2	2.439	1.680	2.677	1.572	2.836	1.572	2.295
			γ_3	2.439	1.680	2.677	1.572	2.836	1.572	2.295
			γ_4	2.371	1.684	2.667	1.558	2.753	1.565	2.267
40	10		γ_1	2.893	1.592	2.719	1.578	2.806	1.315	1.901
			γ_2	2.825	1.578	2.651	1.561	2.736	1.301	1.841
			γ_3	2.825	1.578	2.651	1.561	2.736	1.301	1.841
			γ_4	2.799	1.573	2.625	1.554	2.710	1.295	1.817
80	20		γ_1	2.067	1.153	1.722	1.459	2.104	1.381	2.047
			γ_2	2.083	1.144	1.708	1.450	2.062	1.378	2.034
			γ_3	2.083	1.144	1.908	1.450	2.062	1.378	2.034
			γ_4	2.091	1.141	1.738	1.446	2.045	1.377	2.030
120	30		γ_1	1.431	1.665	3.537	1.588	2.606	1.596	2.420
			γ_2	1.436	1.659	3.442	1.588	2.596	1.595	2.420
			γ_3	1.436	1.659	3.442	1.588	2.596	1.595	2.420
			γ_4	1.439	1.657	3.407	1.588	2.592	1.594	2.421
200	50		γ_1	1.032	1.268	1.593	1.379	1.896	1.000	1.059
			γ_2	1.035	1.268	1.595	1.375	1.874	2.133	1.056
			γ_3	1.035	1.268	1.595	1.375	1.874	2.133	1.056
			γ_4	1.036	1.268	1.596	1.374	1.865	2.131	1.055
20	5	0.9	γ_1	1.755	1.310	1.372	1.032	1.134	1.195	1.163
			γ_2	1.598	1.321	1.422	1.026	1.115	1.186	1.190
			γ_3	1.598	1.321	1.422	1.026	1.115	1.186	1.190
			γ_4	1.544	1.326	1.443	1.023	1.107	1.181	1.193
40	10		γ_1	1.794	1.197	1.627	1.327	2.014	1.134	1.338
			γ_2	1.730	1.184	1.572	1.303	1.914	1.127	1.320
			γ_3	1.730	1.184	1.572	1.303	1.914	1.127	1.320
			γ_4	1.704	1.179	1.551	1.293	1.874	1.124	1.312
80	20		γ_1	2.129	0.882	1.011	1.199	1.452	1.259	1.565
			γ_2	2.130	0.879	1.010	1.195	1.432	1.279	1.538
			γ_3	2.130	0.879	1.128	1.195	1.432	1.464	1.649
			γ_4	2.132	0.877	1.031	1.194	1.424	1.476	1.790
120	30		γ_1	1.195	1.188	1.697	1.372	1.860	1.256	1.455
			γ_2	1.194	1.182	1.651	1.369	1.843	1.260	1.470
			γ_3	1.194	1.182	1.651	1.369	1.843	1.260	1.470
			γ_4	1.194	1.180	1.634	1.368	1.836	1.262	1.476
200	50		γ_1	1.363	1.280	1.611	1.180	1.436	1.503	1.544
			γ_2	1.351	1.282	1.619	1.177	1.424	1.496	1.427
			γ_3	1.351	1.282	1.619	1.177	1.424	1.496	1.753
			γ_4	1.346	1.283	1.623	1.176	1.420	1.494	1.457
20	5	0.7	γ_1	1.428	1.218	1.485	1.037	1.173	1.156	1.080
			γ_2	1.339	1.214	1.427	1.021	1.105	1.159	1.100
			γ_3	1.339	1.214	1.427	1.021	1.105	1.159	1.100
			γ_4	1.308	1.213	1.409	1.014	1.077	1.160	1.107
40	10		γ_1	1.591	1.199	1.468	1.254	1.414	1.591	1.031
			γ_2	1.597	1.185	1.412	1.251	1.429	1.597	1.016
			γ_3	1.597	1.185	1.412	1.251	1.429	1.597	1.016
			γ_4	1.600	1.180	1.390	1.250	1.436	1.600	1.009
80	20		γ_1	1.408	1.031	1.031	1.022	1.093	1.260	1.547
			γ_2	1.416	1.021	1.416	1.020	1.086	1.260	1.545
			γ_3	1.416	1.140	1.416	1.020	1.086	1.260	1.545
			γ_4	1.420	1.037	1.420	1.019	1.084	1.260	1.545
120	30		γ_1	1.032	1.144	1.605	1.128	1.155	1.318	1.694
			γ_2	1.032	1.139	1.552	1.129	1.161	1.325	1.721
			γ_3	1.032	1.139	1.552	1.129	1.161	1.325	1.721
			γ_4	1.032	1.136	1.532	1.130	1.163	1.327	1.734
200	50		γ_1	1.301	1.086	1.211	1.013	1.069	1.095	1.038
			γ_2	1.292	1.088	1.215	1.012	1.066	1.093	1.032
			γ_3	1.292	1.088	1.215	1.012	1.066	1.093	1.032
			γ_4	1.288	1.089	1.217	1.011	1.065	1.092	1.029

Table A.12
RE of the different entropy measures for IEP distribution for Set 2

Set 2										
t^*	d	ρ	Entropy_Type	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	γ_1	1.518	1.137	1.439	1.515	2.120	1.227	1.488
			γ_2	1.474	1.123	1.390	1.492	1.953	1.231	1.562
			γ_3	1.474	1.123	1.390	1.492	1.953	1.231	1.562
			γ_4	1.526	1.118	1.375	1.483	1.890	1.234	1.603
40	10		γ_1	2.281	1.273	1.558	1.557	2.315	1.068	1.235
			γ_2	1.779	1.267	1.555	1.565	2.387	1.058	1.216
			γ_3	1.779	1.267	1.555	1.565	2.387	1.058	1.216
			γ_4	1.731	1.265	1.558	1.569	2.425	1.054	1.213
80	20		γ_1	1.165	1.102	1.312	1.291	1.689	1.387	1.820
			γ_2	1.140	1.097	1.282	1.284	1.674	1.390	1.822
			γ_3	1.140	1.097	1.282	1.284	1.674	1.390	1.822
			γ_4	1.129	1.096	1.271	1.282	1.670	1.392	1.826
120	30		γ_1	1.730	1.448	2.078	1.560	2.416	1.417	1.984
			γ_2	1.764	1.440	2.008	1.567	2.461	1.409	1.989
			γ_3	1.764	1.440	2.008	1.567	2.461	1.409	1.989
			γ_4	1.782	1.438	1.983	1.570	2.483	1.406	1.996
200	50		γ_1	1.295	1.318	1.812	1.572	2.877	1.261	1.523
			γ_2	1.354	1.312	1.780	1.579	2.957	1.256	1.510
			γ_3	1.354	1.312	1.780	1.579	2.957	1.256	1.510
			γ_4	1.381	1.309	1.767	1.583	2.996	1.254	1.505
20	5	0.9	γ_1	1.450	1.179	1.301	1.304	1.615	1.243	1.546
			γ_2	1.418	1.176	1.299	1.318	1.623	1.245	1.625
			γ_3	1.418	1.176	1.299	1.318	1.623	1.245	1.625
			γ_4	1.469	1.176	1.301	1.324	1.626	1.248	1.666
40	10		γ_1	2.034	1.102	1.255	1.268	1.518	1.824	1.604
			γ_2	1.673	1.104	1.260	1.255	1.483	1.898	1.514
			γ_3	1.673	1.104	1.260	1.255	1.483	1.898	1.514
			γ_4	1.667	1.106	1.266	1.250	1.469	1.939	1.480
80	20		γ_1	1.504	1.074	1.307	1.502	2.155	1.140	1.294
			γ_2	1.450	1.071	1.281	1.496	2.144	1.140	1.295
			γ_3	1.450	1.071	1.281	1.496	2.144	1.140	1.295
			γ_4	1.428	1.070	1.271	1.494	2.143	1.140	1.298
120	30		γ_1	1.754	1.163	1.401	1.449	2.086	1.287	1.629
			γ_2	1.760	1.167	1.390	1.454	2.105	1.272	1.607
			γ_3	1.760	1.167	1.390	1.454	2.105	1.272	1.607
			γ_4	1.765	1.170	1.388	1.456	2.116	1.266	1.601
200	50		γ_1	1.259	1.441	1.868	1.254	1.720	1.223	1.676
			γ_2	1.263	1.442	1.869	1.254	1.740	1.222	1.668
			γ_3	1.263	1.442	1.869	1.254	1.740	1.222	1.668
			γ_4	1.265	1.442	1.871	1.254	1.750	1.222	1.666
20	5	0.7	γ_1	1.069	1.023	1.153	1.157	1.201	1.002	1.007
			γ_2	1.006	1.009	1.110	1.147	1.118	1.565	1.117
			γ_3	1.006	1.009	1.110	1.147	1.118	1.565	1.117
			γ_4	1.019	1.003	1.094	1.143	1.082	1.590	1.113
40	10		γ_1	1.869	1.173	1.350	1.056	1.102	1.635	1.080
			γ_2	1.535	1.165	1.342	1.046	1.143	1.749	1.080
			γ_3	1.535	1.165	1.342	1.046	1.143	1.749	1.080
			γ_4	1.525	1.162	1.340	1.041	1.286	1.806	1.080
80	20		γ_1	1.055	1.978	1.951	1.061	1.114	1.299	1.479
			γ_2	1.032	1.984	1.962	1.056	1.107	1.309	1.516
			γ_3	1.032	1.984	1.962	1.056	1.107	1.309	1.516
			γ_4	1.022	1.987	1.967	1.053	1.105	1.313	1.535
120	30		γ_1	1.501	1.046	1.203	1.384	1.586	1.127	1.316
			γ_2	1.506	1.041	1.166	1.376	1.528	1.108	1.275
			γ_3	1.506	1.041	1.166	1.376	1.528	1.108	1.275
			γ_4	1.509	1.039	1.152	1.373	1.503	1.100	1.260
200	50		γ_1	1.616	1.337	1.754	1.189	1.434	1.163	1.267
			γ_2	1.646	1.338	1.753	1.194	1.471	1.165	1.261
			γ_3	1.646	1.338	1.753	1.194	1.471	1.165	1.261
			γ_4	1.659	1.338	1.754	1.197	1.489	1.165	1.258

Table A.13
RE of the different entropy measures for IEP distribution for Set 3

Set 3										
t^*	d	ρ	Entropy_Type	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	γ_1	2.298	1.390	2.040	1.471	2.196	1.738	2.710
			γ_2	2.583	1.368	2.007	1.552	2.802	1.790	2.755
			γ_3	2.583	1.368	2.007	1.552	2.802	1.790	2.755
			γ_4	2.391	1.373	1.975	1.506	2.436	1.759	2.701

(continued on next page)

Table A.13 (*continued*)

Set 3										
40	10		γ_1	2.198	1.548	2.401	1.723	2.841	1.501	2.011
			γ_2	2.605	1.574	2.452	1.771	3.093	1.573	2.363
			γ_3	2.605	1.574	2.452	1.771	3.093	1.573	2.363
			γ_4	2.379	1.558	2.405	1.744	2.934	1.535	2.170
80	20		γ_1	1.870	1.322	1.929	1.426	1.948	1.535	2.173
			γ_2	1.925	1.344	2.081	1.472	2.213	1.545	2.300
			γ_3	1.925	1.344	2.081	1.472	2.213	1.545	2.300
			γ_4	1.893	1.332	1.996	1.448	2.070	1.539	2.227
120	30		γ_1	2.306	1.930	3.817	1.495	2.333	1.829	3.408
			γ_2	2.239	1.964	4.315	1.491	2.348	1.863	3.558
			γ_3	2.239	1.964	4.315	1.491	2.348	1.863	3.558
			γ_4	2.264	1.945	4.031	1.492	2.333	1.844	3.465
200	50		γ_1	1.729	1.547	2.237	1.724	3.015	1.511	2.231
			γ_2	1.579	1.553	2.277	1.714	2.936	1.512	2.195
			γ_3	1.579	1.553	2.277	1.714	2.936	1.512	2.195
			γ_4	1.650	1.549	2.254	1.718	2.964	1.511	2.210
20	5	0.9	γ_1	1.893	1.255	1.633	1.357	1.868	1.426	1.904
			γ_2	2.288	1.224	1.461	1.310	1.766	1.462	1.808
			γ_3	2.288	1.224	1.461	1.310	1.766	1.462	1.808
			γ_4	2.047	1.236	1.520	1.329	1.785	1.442	1.855
40	10		γ_1	1.913	1.411	1.962	1.311	1.831	1.063	1.074
			γ_2	2.171	1.411	1.984	1.397	2.250	1.090	1.168
			γ_3	2.171	1.411	1.984	1.397	2.250	1.090	1.168
			γ_4	2.026	1.409	1.955	1.351	2.015	1.076	1.122
80	20		γ_1	2.468	1.159	1.475	1.536	2.188	1.057	1.158
			γ_2	2.569	1.167	1.563	1.577	2.425	1.046	1.140
			γ_3	2.569	1.167	1.563	1.577	2.425	1.046	1.140
			γ_4	2.510	1.162	1.513	1.555	2.296	1.051	1.146
120	30		γ_1	2.297	1.408	1.957	1.382	1.891	1.430	1.904
			γ_2	2.230	1.407	2.080	1.371	1.859	1.468	2.043
			γ_3	2.230	1.407	2.080	1.371	1.859	1.468	2.043
			γ_4	2.256	1.406	2.005	1.376	1.870	1.448	1.966
200	50		γ_1	1.126	1.322	1.779	1.347	1.968	1.504	2.184
			γ_2	1.089	1.315	1.773	1.344	1.912	1.498	2.163
			γ_3	1.089	1.315	1.773	1.344	1.912	1.498	2.163
			γ_4	1.107	1.318	1.774	1.345	1.933	1.501	2.170
20	5	0.7	γ_1	1.174	1.024	1.220	1.203	1.415	1.271	1.408
			γ_2	1.283	1.041	1.331	1.205	1.509	1.272	1.270
			γ_3	1.283	1.041	1.331	1.205	1.509	1.272	1.270
			γ_4	1.223	1.030	1.257	1.202	1.448	1.271	1.339
40	10		γ_1	1.576	1.214	1.440	1.123	1.072	1.014	1.056
			γ_2	1.644	1.240	1.495	1.177	1.190	1.020	1.060
			γ_3	1.644	1.240	1.495	1.177	1.190	1.020	1.060
			γ_4	1.599	1.226	1.463	1.151	1.137	1.017	1.056
80	20		γ_1	1.275	1.098	1.264	1.107	1.127	1.345	1.531
			γ_2	1.311	1.103	1.310	1.150	1.317	1.329	1.476
			γ_3	1.311	1.103	1.310	1.150	1.317	1.329	1.476
			γ_4	1.291	1.100	1.285	1.128	1.220	1.337	1.502
120	30		γ_1	1.604	1.127	1.361	1.350	1.658	1.288	1.626
			γ_2	1.622	1.145	1.529	1.351	1.689	1.324	1.731
			γ_3	1.622	1.145	1.529	1.351	1.689	1.324	1.731
			γ_4	1.611	1.135	1.437	1.350	1.670	1.305	1.674
200	50		γ_1	1.405	1.245	1.586	1.196	1.392	1.330	1.610
			γ_2	1.330	1.237	1.556	1.191	1.358	1.332	1.631
			γ_3	1.330	1.237	1.556	1.191	1.358	1.332	1.631
			γ_4	1.365	1.240	1.569	1.193	1.372	1.331	1.620

Table A.14
RE of the different entropy measures for IEP distribution for Set 4

Set 4										
t^*	d	ρ	Entropy_Type	ML	MPS	KM	LS	WLS	AD	CVM
20	5	1	γ_1	2.241	1.267	1.728	1.516	2.365	1.750	2.892
			γ_2	2.561	1.278	1.813	1.612	3.139	1.830	3.218
			γ_3	2.561	1.278	1.813	1.612	3.139	1.830	3.218
			γ_4	2.363	1.269	1.749	1.559	2.688	1.784	3.015
40	10		γ_1	2.265	1.520	2.180	1.772	2.888	1.447	1.965
			γ_2	2.648	1.544	2.245	1.801	3.009	1.502	2.220
			γ_3	2.648	1.544	2.245	1.801	3.009	1.502	2.220
			γ_4	2.439	1.531	2.201	1.785	2.934	1.473	2.082
80	20		γ_1	1.739	1.269	1.828	1.468	2.164	1.507	2.138
			γ_2	1.810	1.288	1.947	1.513	2.469	1.517	2.238

(continued on next page)

Table A.14 (*continued*)

Set 4										
			γ_3	1.810	1.288	1.947	1.513	2.469	1.517	2.238
			γ_4	1.771	1.278	1.881	1.489	2.306	1.511	2.181
120	30		γ_1	2.228	1.841	3.464	1.657	2.798	1.766	3.105
			γ_2	2.180	1.870	3.852	1.652	2.810	1.794	3.200
			γ_3	2.180	1.870	3.852	1.652	2.810	1.794	3.200
			γ_4	2.198	1.854	3.635	1.654	2.798	1.779	3.142
200	50		γ_1	1.665	1.557	2.199	1.625	2.716	1.522	2.175
			γ_2	1.540	1.566	2.251	1.618	2.671	1.524	2.153
			γ_3	1.540	1.566	2.251	1.618	2.671	1.524	2.153
			γ_4	1.600	1.561	2.223	1.621	2.689	1.523	2.162
20	5	0.9	γ_1	1.815	1.183	1.499	1.448	1.977	1.537	2.470
			γ_2	2.057	1.189	1.522	1.470	2.202	1.621	2.827
			γ_3	2.057	1.189	1.522	1.470	2.202	1.621	2.827
			γ_4	1.910	1.184	1.498	1.457	2.067	1.574	2.614
40	10		γ_1	1.777	1.295	1.674	1.511	2.209	1.037	1.075
			γ_2	1.948	1.297	1.709	1.596	2.571	1.058	1.147
			γ_3	1.948	1.297	1.709	1.596	2.571	1.058	1.147
			γ_4	1.852	1.295	1.682	1.552	2.377	1.047	1.110
80	20		γ_1	2.107	1.127	1.482	1.566	2.262	1.158	1.296
			γ_2	2.210	1.134	1.558	1.611	2.497	1.152	1.290
			γ_3	2.210	1.134	1.558	1.611	2.497	1.152	1.290
			γ_4	2.154	1.130	1.515	1.588	2.373	1.155	1.291
120	30		γ_1	2.217	1.350	1.856	1.445	2.067	1.400	1.844
			γ_2	2.176	1.353	1.971	1.432	2.017	1.430	1.938
			γ_3	2.176	1.353	1.971	1.432	2.017	1.430	1.938
			γ_4	2.191	1.350	1.904	1.438	2.038	1.415	1.886
200	50		γ_1	1.187	1.373	1.772	1.401	2.068	1.569	2.407
			γ_2	1.159	1.369	1.765	1.399	2.025	1.571	2.431
			γ_3	1.159	1.369	1.765	1.399	2.025	1.571	2.431
			γ_4	1.172	1.371	1.767	1.400	2.043	1.570	2.416
20	5	0.7	γ_1	1.255	1.001	1.164	1.205	1.453	1.365	1.611
			γ_2	1.412	1.019	1.255	1.206	1.566	1.393	1.587
			γ_3	1.412	1.019	1.255	1.206	1.566	1.393	1.587
			γ_4	1.333	1.009	1.199	1.204	1.494	1.378	1.598
40	10		γ_1	1.525	1.165	1.348	1.149	1.152	1.525	1.121
			γ_2	1.626	1.190	1.388	1.198	1.266	1.626	1.189
			γ_3	1.626	1.190	1.388	1.198	1.266	1.626	1.189
			γ_4	1.570	1.177	1.368	1.174	1.212	1.570	1.153
80	20		γ_1	1.229	1.167	1.167	1.025	1.032	1.275	1.528
			γ_2	1.260	1.187	1.260	1.050	1.138	1.269	1.520
			γ_3	1.260	1.187	1.260	1.050	1.138	1.269	1.520
			γ_4	1.243	1.175	1.243	1.037	1.084	1.271	1.522
120	30		γ_1	1.702	1.167	1.508	1.278	1.655	1.279	1.582
			γ_2	1.739	1.190	1.703	1.275	1.670	1.316	1.697
			γ_3	1.739	1.190	1.703	1.275	1.670	1.316	1.697
			γ_4	1.718	1.177	1.597	1.276	1.660	1.297	1.635
200	50		γ_1	1.494	1.306	1.651	1.128	1.294	1.294	1.615
			γ_2	1.443	1.300	1.628	1.124	1.270	1.290	1.607
			γ_3	1.443	1.300	1.628	1.124	1.270	1.290	1.607
			γ_4	1.467	1.303	1.638	1.126	1.281	1.292	1.610

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