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Inference of dynamic weighted cumulative residual entropy for Burr XII distribution based on progressive censoring

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Abstract

The dynamic weighted cumulative residual (DWCR) entropy is regarded as an additional measure of uncertainty related to the residual lifetime function in several disciplines, including survival analysis and reliability. This article presents the DWCR formula based on Havarda and Charvat. This measurement is called the DWCR Havarda and Charvat entropy (DWCRHCE). This work uses progressive Type II censoring to investigate the implications of DWCR Tsallis entropy (DWCRTE), DWCR Rényi entropy (DWCRRE), and DWCRHCE for the Burr XII distribution. Both classical and Bayesian methods are used to derive the estimators of these entropy metrics. Assuming independent gamma priors, we get the Bayes estimator of the suggested measures. Due to the lack of explicit forms, the Metropolis-Hastings approach was offered to determine the Bayes estimates for symmetric and asymmetric loss functions. To determine the efficacy of the suggested estimating techniques, several simulations were run for different censoring schemes. The simulation analysis leads us to the conclusion that, under a precautionary loss function followed by a linear exponential loss function, the Bayesian estimates of DWCRTE are generally more effective than the DWCRHCE or DWCRRE. Compared to maximum likelihood estimates, Bayesian estimates are preferred for different metrics. After that, a detailed explanation of the process is provided by looking at real data. The analysis of real-world data, specifically the Shasta reservoir water capacity data, aligns with the findings from simulated data. Notably, these findings have crucial implications for effective water resource management decisions.

Key words: Burr XII distribution, dynamic weighted cumulative residual entropy, Bayesian estimators, precautionary loss function.
Mathematical Subject Classification: 62F10.

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1. Introduction

1.1. Progressive Type II censoring

Censorship is common in many fields, including pharmacology, social economics, and engineering, particularly in reliability and survival analysis (Wang & Gui, 2021). Due to time and cost constraints, it is difficult to completely observe the sample data in actual production. Even though they have been thoroughly examined, conventional censoring systems such as Type-I, Type-II, and hybrid schemes are inflexible, in that units cannot be removed arbitrarily. Cohen (1963) proposed the progressive censoring scheme (PCS) in order to overcome this restriction. In the PCS, units are removed from the experiment at different time points, with the number of units eliminated at each time point specified in advance.

The progressive type-II censoring (PT-IIC) method is one of the most popular censoring schemes. Here, we provide its description. Assume that m failures will be noticed when n identical units are put through a test. At the moment of the initial failure ($y_{(1)}$), the number r_1 of the surviving units ($n-1$) is randomly selected and removed from the experiment. At the second failure ($y_{(2)}$), the number r_2 of the surviving units ($n-r_1-2$) is randomly selected and removed from the experiment, and so on, until the m^{th} failure ($y_{(m)}$) occurs, at which point all the remaining $n-m-r_1-r_2-\cdots-r_{m-1}$ units are removed. Thus, the PCS includes m observed samples of failure $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(m)}$, and survival items $\boldsymbol{r} = (r_1, r_2, \cdots, r_m)$ such that $n = m + r_1 + \cdots + r_m$. Note that in the PCS, $\boldsymbol{r} = (r_1, r_2, \cdots, r_m)$ is prefixed. Balakrishnan and Aggrawala (2000) offered historical context and a comprehensive overview of PCS. Notably, the following special cases can be noticed:

- Classical Type-II censoring: This occurs when $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n-m$.
- Complete sample: This occurs when $m = n$ and $r_i = 0$, $i = 1, 2, \cdots, n$.

1.2. Entropy measures

Shannon (1948) proposed the concept of entropy as a metric for quantifying uncertainty. Nowadays, the fields of economics, physics, telecommunications, communication theory, and reliability have given this criterion significant consideration. One parameter generalization of the Shannon entropy that may be applied as a randomness metric is the Rényi entropy. Numerous disciplines, including biology, genetics, electrical engineering, computer science, economics, chemistry and physics use Rényi entropy in their work. The entropy function as an extension of the Shannon entropy was first presented by Rényi (1961), followed by Havrda and Charvat (1967). The Rényi and Havrda and Charvat entropy of order c are defined respectively by the following expressions:

$$\boldsymbol{R(c)} = \frac{\mathbf{1}}{\mathbf{1-c}} \boldsymbol{log}\left(\int_{-\infty}^{\infty} (\boldsymbol{f(y)})^{\boldsymbol{c}}\, \boldsymbol{dy}\right); \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1},$$

$$\boldsymbol{H(c)} = \left(\mathbf{2^{1-c}} - \mathbf{1}\right)^{-\mathbf{1}} \left(\int_{-\infty}^{\infty} (\boldsymbol{f(y)})^{\boldsymbol{c}} - \mathbf{1}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1}.$$

Tsallis (1988) also presented the ideas of Tsallis entropy as a measure of quantifying randomness. The entropy can be used to determine the level of uncertainty associated with a random observation. Tsallis later utilized its distinctive characteristics and situated it within a physical framework. This metric is designated for any continuous random variable of order *c*, where and it is defined as follows:

$$\boldsymbol{T(c)} = \frac{\mathbf{1}}{(\boldsymbol{c}-\mathbf{1})}\left(\mathbf{1} - \int_{-\infty}^{\infty} \boldsymbol{f^c(y)dy}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1}.$$

Recently, entropy measure estimation utilizing different statistical distributions and sampling strategies has been studied by several writers (see, for example, Baratpour et al., 2007; Abo-Eleneen, 2011; Cho et al., 2015; Lee, 2017; Hassan and Zaky, 2019, 2021; Helmy et al., 2021; Hassan et al., 2022; Helmy et al., 2023; and Hassan at al., 2024b).

Different measures of uncertainty for probability distributions have attracted many writers, especially in works related to reliability analysis and survival. In light of Rényi's entropy's utility, Sunoj and Linu (2012) presented cumulative residual Rényi entropy (CRRE) of order *c* as below:

$$\bar{\boldsymbol{R}}(\boldsymbol{c}) = \frac{\mathbf{1}}{(\mathbf{1}-\boldsymbol{c})}\boldsymbol{log}\left(\int_{t}^{\infty} (\overline{\boldsymbol{F}}(\boldsymbol{y}))^{\boldsymbol{c}}\boldsymbol{dy}\right); \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1},$$

where $\bar{F}(y) = 1 - F(y)$ is the survival function (SF). Additionally, Sunoj and Linu (2012) examined the primary characteristics of the dynamic version of the CRRE (DCRRE), which is extended to the residual lifetime $Y_t = (Y - t \mid Y > t)$ based on survival function, rather than using probability density function (PDF), and found it to be beneficial for reliability modeling. The DCRRE of order *c* is given by:

$$\overline{\boldsymbol{R}}^{*}(\boldsymbol{c}) = (\mathbf{1}-\boldsymbol{c})^{-\mathbf{1}}\,\boldsymbol{log}\left(\frac{\mathbf{1}}{(\overline{\boldsymbol{F}}(\boldsymbol{t}))^{\boldsymbol{c}}}\int_{t}^{\infty}(\overline{\boldsymbol{F}}(\boldsymbol{y}))^{\boldsymbol{c}}\boldsymbol{dy}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1}. \tag{1}$$

Sati and Gupta (2015) established the DCR Tsallis entropy (DCRTE) as follows:

$$\overline{\boldsymbol{T}}^{*}(\boldsymbol{c}) = (\boldsymbol{c}-\mathbf{1})^{-\mathbf{1}}\left(\mathbf{1} - (\overline{\boldsymbol{F}}(\boldsymbol{t}))^{-\boldsymbol{c}}\int_{t}^{\infty}(\overline{\boldsymbol{F}}(\boldsymbol{y}))^{\boldsymbol{c}}\boldsymbol{dy}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1},$$

In the literature, some statistical inferences based on the above dynamic entropy measures and their related works have been considered by several authors. For the Pareto distribution, Bayesian estimates (BEs) of the DCR entropy under various sampling conditions have been examined by several researchers (see Renjini et al., 2016a, 2016b, and 2018; and Ahmadini et al., 2020). The Lindley distribution's BE of DCRRE was examined by Almarashi et al. (2021). Al-Babtain et al. (2021) supplied the maximum likelihood estimates (MLEs) and BEs of the DCRRE for the Lomax distribution. Mohamed (2022) used generalized order statistics to study DCRTE and cumulative residual Tsallis entropy. The BEs of the DCRTE for the moment exponential distribution were recently studied by Alyami et al. (2023). For more recent studies, refer to Kayal and Balakrishnan (2023), Nair and Sathar (2024) and Smitha et al. (2024).

Weighted distributions provide a valuable tool for modeling statistical data in situations where standard distributions may not accurately capture the underlying characteristics of the data. Guiasu (1986) applied weighted entropy in order to balance the degree of homogeneity and information contained in a data partition into classes. The idea of weighted distributions was used in a number of domains, such as biostatistics (Wang, 1996), reliability modeling (Navarro et al., 2001) and renewal theory (Sunoj and Mayi, 2006). In the context of theoretical neurobiology, uncertainty measures based on the concept of weighted entropy were explored by Belis and Guiasu (1968). Di Crescenzo and Longobardi (2006) extended the concept of weighted entropy to residual and past lifetimes, introducing weighted residual and past entropies. This work builds upon previous research by Belzunce et al. (2004) and Nanda and Paul (2006), which characterized distribution functions using weighted dynamic measures. Misagh and Yari (2011) further investigated this concept by studying the weighted differential information measure for two-sided truncated random variables. Sunoj and Linu (2012) introduced a new measure of uncertainty based on the length-biased weighted function and called it dynamic weighted CRRE (DWCRRE). Based on the SF given in Equation (1), the DWCRRE is as follows:

$$\boldsymbol{R}^{*}(\boldsymbol{c}) = (\mathbf{1}-\boldsymbol{c})^{-\mathbf{1}}\, \boldsymbol{log}\left(\left(\overline{\boldsymbol{F}}(\boldsymbol{t})\right)^{-\boldsymbol{c}} \int_{\boldsymbol{t}}^{\infty} \boldsymbol{y}\left(\overline{\boldsymbol{F}}(\boldsymbol{y})\right)^{\boldsymbol{c}} \boldsymbol{dy}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1}, \tag{2}$$

where y is the length-biased weighted function. The dynamic weighted CRTE (DWCRTE), introduced by Khammar and Jahanshahi (2018), is another significant weighted metric. It is defined as:

$$\boldsymbol{T}^{*}(\boldsymbol{c}) = (\boldsymbol{c}-\mathbf{1})^{-\mathbf{1}}\left(\mathbf{1} - \left(\overline{\boldsymbol{F}}(\boldsymbol{t})\right)^{-\boldsymbol{c}} \int_{\boldsymbol{t}}^{\infty} \boldsymbol{y}\left(\overline{\boldsymbol{F}}(\boldsymbol{y})\right)^{\boldsymbol{c}} \boldsymbol{dy}\right); \quad \boldsymbol{c} > \mathbf{0}, \boldsymbol{c} \neq \mathbf{1}. \tag{3}$$

Using Type II right-censored data, a weighted version of CRTE and DCRTE was created by Khammar and Jahanshahi (2018), and many of its reliability properties.

1.3. Work Motivation

As far as we are aware, no research so far has taken into account the PT-IIC for dynamic weighted cumulative residual in entropy measurements. Therefore, our research question is: "How to find the estimate of dynamic weighted cumulative residual entropy measures under PT-IIC for the Burr XII distribution (BXIID)?". We decided to investigate this topic because of the significance of the BXIID and its widespread application in numerous sectors (as shown in Section 2). Our work involves the following steps:

- The DWCRE based on the Havrda and Charvat measure is defined following the idea of Sunoj and Linu (2012). This measure is called the dynamic weighted cumulative residual Havrda and Charvat entropy (DWCRHCE).
- The estimators for the DWCRRE, DWCRTE, and DWCRHCE are derived using both Bayesian and non-Bayesian techniques. Both symmetric and asymmetric loss

functions yield Bayesian estimators for the suggested measures. Also, because there are no explicit forms for the BEs of various measures, we use the Markov chain Monte Carlo (MCMC) approach to approximate the estimates.

- Simulation studies are employed to evaluate and contrast the precision of various approximations about their mean squared error (MSE) and average of estimates. For illustration reasons, application to real data is shown.

The rest of the paper is organized in the following way: Section 2 presents a model description. Dynamic weighted cumulative residual entropy expressions are determined in Section 3. The PT-IIC and maximum likelihood estimation are used in Section 4. The BE of the DWCRE for the BXIID under symmetric and asymmetric loss functions is presented in Section 5. Section 6 provides an example of a real-data application. A simulation study is provided in Section 7. Based on the outcomes of the numerical studies, the paper draws a few conclusions in the last section.

2. Model Description

The Burr distribution is a versatile family that covers several widely used distributions as limiting asymptotic approximations, and it comprises a broad range of distribution shapes. A significant amount of the curve shape properties in the Pearson family are covered by correctly selecting the parameters of the Burr distribution, as Burr (1942) showed. Because its shape parameter generates a variety of forms that are excellent fits for different data, the BXIID has been used in research related to medical, business, chemical engineering, quality control and reliability. Evans and Simons (1975), Wingo (1993) and Gupta et al. (1996) are a few references to consult for a detailed explanation of such circumstances. The PDF and SF of the BXIID have the following specifications:

$$f(y) = \delta\lambda y^{\delta-1}\left(1 + y^{\delta}\right)^{-(\lambda+1)}, y > 0, \tag{4}$$

$$\boldsymbol{\bar{F}(y)} = \left(\mathbf{1} + \boldsymbol{y^{\delta}}\right)^{-\boldsymbol{\lambda}}, \boldsymbol{y} > \mathbf{0}, \tag{5}$$

where$\delta >$ and$\lambda > 0$ are shape parameters. Recently, numerous scholars have conducted extensive research on the estimation utilizing the BXIID. Estimation in step-stress partially accelerated life tests for the BXIID using Type I censoring was covered by Abd-Elfattah et al. (2008). Works on BXIID inferences under the PCS were discussed by, for example, Mousa and Jaheen (2002), Wu and Yu (2005), Soliman (2005), Li et al. (2007) and Hassan et al. (2024a). According to Panahi and Sayyareh (2014), statistical inference and prediction about BXIID parameters based on a Type II censored sample were covered. On the basis of the competing risk model, Qin and Gui (2020) derived the MLE and BE of BXIID parameters.

3. Dynamic Weighted Cumulative Residual Entropy Expressions

Inspired by the DWCRRE entropy developed by Sunoj and Linu (2012) and the DWCRTE presented by Khammar and Jahanshahi (2018), we introduce two novel information measures: the DCR Havrda and Charvat entropy (DCRHCE) and the second measure is the DWCRHCE. This measure is based on the cumulative residual entropy originally proposed by Rao et al. (2004), and the dynamic cumulative residual entropy proposed by Asadi and Zohrevand (2007) and Sati and Gupta (2015).

Definition: The DCRHCE and DWCRHCE of a random variable Y of order c are defined by:

$$\boldsymbol{H(c) = \left(2^{1-c}-1\right)^{-1}\left((\bar{F}(t))^{-c}\int_t^{\infty}(\bar{F}(y))^c - 1\right); c>0, c\neq 1,}$$

$$\boldsymbol{H^{*}(c) = \left(2^{1-c}-1\right)^{-1}\left((\bar{F}(t))^{-c}\int_t^{\infty}y(\bar{F}(y))^c - 1\right); c>0, c\neq 1,} \tag{6}$$

where y, is the length-biased weighted function. Now, the expressions of DWCRRE, DWCRTE, and DWCRHCE for the BXIID are obtained. To do so, we first obtain the following integral by using the SF given in Equation (5)

$$I = \int_t^{\infty} y(\bar{F}(y))^c dy = \int_t^{\infty} y\left(1+y^{\delta}\right)^{-\lambda c} dy. \tag{7}$$

Using the transformation $z = y^{\delta}$, $y = (z)^{\frac{1}{\delta}}$, $dy = \delta^{-1}z^{\frac{1}{\delta}-1}dz$in Equation (7) yields:

$$I = \int_{t^{\delta}}^{\infty} z^{\frac{1}{\delta}}(1+z)^{-\lambda c}\frac{1}{\delta}z^{\frac{1}{\delta}-1}dz = \frac{1}{\delta}\int_{t^{\delta}}^{\infty}(1+z)^{-\lambda c}z^{\frac{2}{\delta}-1}dz.$$

Use the transformation $x = (1+z)^{-1}$, then $z = \frac{(1-x)}{x}$, and$dz = \frac{-dx}{x^2}$, thus the integral I is as follows:

$$I = \frac{1}{\delta}\int_0^{(1+t^{\delta})^{-1}} x^{\lambda c}\left(\frac{1-x}{x}\right)^{\frac{2}{\delta}-1}\frac{dx}{x^2} = \frac{1}{\delta}\int_0^{(1+t^{\delta})^{-1}} x^{\lambda c-\frac{2}{\delta}-1}(1-x)^{\frac{2}{\delta}-1}dx = \frac{1}{\delta}B\left(\frac{2}{\delta}, \lambda c - \frac{2}{\delta}, \left(1+t^{\delta}\right)^{-1}\right), \tag{8}$$

where $B(a,b,x) = \int_0^x y^{a-1}(1-y)^{b-1}dy$ is the incomplete beta function. Now the formula of the DWCRRE is obtained by inserting Equation (8) in Equation (2), in the following way:

$$R^{*}(c) = \frac{1}{(1-c)}log\left(\frac{1}{\delta\left(1+t^{\delta}\right)^{-\lambda c}}B\left(\frac{2}{\delta}, \lambda c - \frac{2}{\delta}, \left(1+t^{\delta}\right)^{-1}\right)\right). \tag{9}$$

Also, expressions of DWCRTE and DWCRHCE measures are obtained by inserting Equation (8) in Equations (3) and (6), respectively, as below:

$$T^{*}(c) = \frac{1}{(c-1)}\left[1 - \frac{1}{\delta\left(1+t^{\delta}\right)^{-\lambda c}}\left(B\left(\frac{2}{\delta}, \lambda c - \frac{2}{\delta}, \left(1+t^{\delta}\right)^{-1}\right)\right)\right], \tag{10}$$

$$H^{*}(c) = \frac{1}{(2^{1-c}-1)}\left[\frac{1}{\delta\left(1+t^{\delta}\right)^{-\lambda c}}\left(B\left(\frac{2}{\delta}, \lambda c - \frac{2}{\delta}, \left(1+t^{\delta}\right)^{-1}\right)\right) - 1\right]. \tag{11}$$

Note that expressions (9), (10) and (11) are a function of parameters δ, λ, and c.

4. Maximum Likelihood Estimators

Here, the MLEs of DWCRRE, DWCRTE, and DWCRHCE of the BXIID, based on PT-IIC samples, are obtained. Assume that $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(m)}$be the PT-IIC of size m from a sample of size n taken from density given in Equation (4) and SF given in Equation (5) with censoring scheme $r_1, r_2, \ldots, r_m$. Based on the PT-IIC sample, the likelihood function reads:

$$L(\delta,\lambda) = D\prod_{i=1}^{m} f(y_{(i)})\left[1-F(y_{(i)})\right]^{r_i} == D\delta^m\lambda^m \prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-\lambda(r_i+1)-1},$$

where $\boldsymbol{D = n(n-r_1-1)(n-r_1-r_2-2)\ldots n-m+1-\sum_{i=1}^{m-1} r_i}$. Additionally, the corresponding log-likelihood function, say ℓ^*, is

$$\boldsymbol{\ell^* \propto m\,ln(\delta) + m\,ln(\lambda) + (\delta-1)\sum_{i=1}^{m} ln(y_{(i)}) - \sum_{i=1}^{m}[\lambda(r_i+1)+1]\,ln\left(1+y_{(i)}^{\delta}\right)}.$$

Then, the first derivative of the log- likelihood function, with respect to the parameters δ and λ, are as follows:

$$\frac{\partial\ell^*}{\partial\delta} = \frac{m}{\delta} + \sum_{i=1}^{m} ln y_{(i)} \quad - \sum_{i=1}^{m} \frac{[\lambda(r_i+1)+1]\, ln y_{(i)}}{\left(1+y_{(i)}^{-\delta}\right)}, \tag{12}$$

and

$$\boldsymbol{\frac{\partial\ell^*}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m}(r_i+1)\,ln\left(1+y_{(i)}^{\delta}\right)}. \tag{13}$$

The MLEs of δ and $\lambda$are determined by solving Equations (12) and (13) after simultaneously setting them to zero by using a numerical technique such as the Newton-Raphson method, to get $\hat{\delta}_{ML}$and $\hat{\lambda}_{ML}$. Based on the invariance property, the MLEs of DWCRRE, DWCRTE, and DWCRHCE are obtained by inserting$\hat{\delta}_{ML}$and $\hat{\lambda}_{ML}$ in Equations (9), (10), and (11).

5. Bayesian Estimator

This part covers the BE of $T^*(c)$, $R^*(c)$ and $H^*(c)$ under both symmetric and asymmetric loss functions for the two-parameter BXIID. It is assumed that the prior of parameters δ and λ have independent gamma priors. The joint prior distribution can be written as:

$$\boldsymbol{\pi_0(\delta,\lambda) \propto \delta^{b_1-1}\lambda^{b_2-1}e^{-(\delta a_1+\lambda a_2)}},$$

where the hyper-parameters a_1 ,a_2 ,b_1, and b_2 are known and non-negative. The gamma prior was chosen for its flexibility in modeling a wide range of prior beliefs. They may take on a wide range of forms based on hyper-parameters. The joint posterior for parameters, denoted by $\pi(\delta,\lambda)$, is

$$\boldsymbol{\pi(\delta,\lambda) = W^{-1}\delta^{m+b_1-1}\lambda^{m+b_2-1}e^{-(\delta a_1+\lambda a_2)}\prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-[\lambda(r_i+1)+1]}},$$

where $\boldsymbol{W = \int_0^\infty\int_0^\infty \delta^{m+b_1-1}\,\lambda^{m+b_2-1}e^{-(\delta a_1+\lambda a_2)}\prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-[\lambda(r_i+1)+1]} d\delta d\lambda}$.

So, the conditional posterior distribution of the unknown parameters δ and λ, is given, respectively, by:

$$\boldsymbol{\pi^*\left(\delta\,\middle|\,\lambda,\underline{y}\right) \propto \delta^{m+b_1-1}e^{-\delta\left(a_1+\sum_{i=1}^{m} ln y_{(i)}\right)-\sum_{i=1}^{m}[\lambda(r_i+1)+1]\,ln\left(1+y_{(i)}^{\delta}\right)}}, \tag{14}$$

and

$$\boldsymbol{\pi^{**}\left(\lambda\,\middle|\,\delta,\underline{y}\right)} \propto \text{Gamma}\left(m+b_2, a_2+\sum_{i=1}^{m}(r_i+1)\,ln(1+y_{(i)}^{\delta})\right). \tag{15}$$

The Bayesian estimators of $g^{\bullet}(c)$, under squared error loss function (SLF), linear exponential loss function (LLF), and precautionary loss function (PRLF) are obtained in the following way:

$$\hat{g}^{\bullet}_{SLF}(c) = E\left[g^{\bullet}(c)\,\middle|\,\underline{y}\right] = W^{-1}\int_0^{\infty}\int_0^{\infty} g^{\bullet}(c)\delta^{m+b_1-1}\lambda^{m+b_2-1}e^{-(\delta a_1+\lambda a_2)}\prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-[\lambda(r_i+1)+1]}d\delta d\lambda, \quad (16)$$

$$\hat{g}^{\bullet}_{LLF}(c) = \frac{-1}{\tau}ln\left[E\left(e^{-\tau g^{\bullet}(c)}\,\middle|\,\underline{y}\right)\right] = W^{-1}\int_0^{\infty}\int_0^{\infty}\delta^{m+b_1-1}\lambda^{m+b_2-1}e^{-(\tau g^{\bullet}(c)+\delta a_1+\lambda a_2)}\prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-[\lambda(r_i+1)+1]}d\delta d\lambda, \quad (17)$$

and

$$\hat{g}^{\bullet}_{PRLF}(c) = \left[E\left((g^{\bullet}(c))^2\,\middle|\,\underline{y}\right)\right]^{0.5} = W^{-1}\int_0^{\infty}\int_0^{\infty}(g^{\bullet}(c))^2\delta^{m+b_1-1}\lambda^{m+b_2-1}e^{-(\delta a_1+\lambda a_2)}\prod_{i=1}^{m} y_{(i)}^{\delta-1}\left(1+y_{(i)}^{\delta}\right)^{-[\lambda(r_i+1)+1]}d\delta d\lambda, \quad (18)$$

where $g^{\bullet}(c) = R^{*}(c)$ to obtain the DWCRRE, $g^{\bullet}(c) = T^{*}(c)$ to calculate the DWCRTE, and $g^{\bullet}(c) = H^{*}(c)$ to produce the DWCRHCE. The Gibbes sampler, Metropolis-Hastings (M-H), and random walk Metropolis algorithms are used to generate the MCMC samples from the posterior density functions (14) and (15), respectively, because integrals (16)–(18) do not take a closed form. As a result, the BEs of δ and λ under the SLF, LLF, and PRLF are calculated from their posteriors as the mean of the simulated samples. The M-H algorithm is one of the most famous subclasses of the MCMC method in Bayesian literature. It is used to simulate the deviates from the posterior density and produce good approximate results. The M-H algorithm uses an acceptance/rejection rule to converge to the target distribution. The initial values of the unknown parameters (δ, λ) must be specified, along with a suggested distribution, in order to implement the M-H algorithm for the DWCRTE, DWCRRE, and the DWCRHCE of the BXIID. A normal distribution will be used to calculate the proposal distribution, i.e., $h(\delta'|\delta)$ and $h(\lambda'|\lambda) \equiv N(\theta, \sigma_{\theta}^2)$, where $\theta \equiv (\delta, \lambda)$ and σ_{θ}^2 is the variance-covariance matrix (Va-CoM) for the MLEs of (δ, λ). The MLE for θ is taken into account, that is, $\theta^{(0)} = \hat{\theta}_{MLE}$. Asymptotic Va-CoM, say $I^{-1}(\hat{\theta}_{MLE})$, where $I(.)$ is the Fisher information matrix, and is assumed to represent the choice of σ_{θ}^2. First, the M-H algorithm employs the steps mentioned below to extract a sample from the posterior density given by Equations (14) and (15).

Step 1: Set the initial value of θ as $\theta^{(0)} = \hat{\theta}_{MLE}$.

Step 2: For i =1, 2, 3,…, M repeat the following steps:

2.1: Set $\theta = \theta^{(i-1)}$.

2.2: Generate λ_1^{i} from Gamma$(m + b_2, a_2 + \sum_{i=1}^{m}\lambda(r_i + 1)\, ln(1 + y_{(i)}^{\delta}))$.

2.3: Create a new candidate parameter value using $N(\theta, S_{\theta})$.

2.4: Compute the formula $\phi = \frac{\pi(\theta'|y)}{\pi(\theta|y),}$ where $\pi(.)$ is the posterior density of Equations (14) and (15).

2.5: Create a sample u from the uniform U (0,1) distribution.

2.6: Accept or reject the new candidate θ'. $\begin{cases} If u \leq \phi put \theta^{(i)} = \theta' \\ elsewhere put \theta^{(i)} = \theta. \end{cases}$

Step 3: Obtain the Bayesian estimator of θ and compute the DWCRTE, DWCRRE, and DWCRHCE functions of $T^*(c)$, $R^*(c)$ and $H^*(c)$ with respect to the loss functions as follows:

$$\widehat{H}^*(c) = \frac{1}{M-Q}\sum_{i=Q+1}^{M} H^*\left(c,\theta^{(i)}\right), \widehat{R}^*(c) = \frac{1}{M-Q}\sum_{i=Q+1}^{M} R^*\left(c,\theta^{(i)}\right), \widehat{T}^*(c) = \frac{1}{M-Q}\sum_{i=Q+1}^{M} T^*\left(c,\theta^{(i)}\right),$$

where Q is the number of samples that have been burned. Ultimately, the estimates of DWCRTE, DWCRRE, and DWCRHCE are obtained by subtracting 2,000 burn-in samples from the 10,000 samples that the posterior density produced.

6. Real Data Analysis

The data set represents the monthly water capacity data from the Shasta reservoir in California, USA, and was taken for the month of February from 1991 to 2010 (http://cdec.water.ca.gov/reservoir_map.html). The maximum capacity of the reservoir is 4552000 AF, and the data set was transformed to the interval [0, 1] (for more details, see Nadar et al., 2013). In the first step, it should be checked if the BXIID is well fitted to this data. By fitting BXIID, the Kolmogorov-Smirnov distance and associated p-value are 7.7852, 7.8451, 0.22479 and 0.2274, respectively. The p-values show that BXIID yields suitable fits for the given dataset. In Figure 1, the empirical distribution functions and histogram are provided.

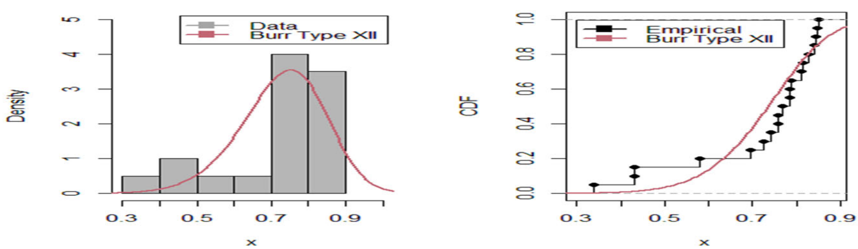

Figure 1. The histogram (left) and the empirical distribution function (right) for a given dataset.

Source: created by researchers utilizing the R programming language.

The number of stages in the PT-IIC scheme is assumed to be m = 12 and the removed items r_i are assumed as in the following scheme (Sch.):

S_1: $(8,0^{*11})$, S_2 : $(4,0^{*10}, 4)$, S_3 : $(0^{*2}, 1^{*8}, 0^{*2})$, and S_4: $(0^{*11}, 8)$.

The complete case also considers instances where $n = m = 2$ and $r_i = 0$, $i = 1, 2, \ldots, n$. Table 1 uses the PT-IIC to compute MLEs based on the produced data for each censoring strategy. Then employed to compute $T^*(c)$, $R^*(c)$ and $H^*(c)$ given t and c, where t = 0.1, 0.2, and c = 1.5, 2.5. For the BEs, the M-H algorithm will be used under different loss functions in the case of a uniform prior, where $a_1 = a_2 = b_1 = b_2 = 0.001$. Different loss functions, including SLF, LLF-1($\tau = 0.5$), LLF-2($\tau = -0.5$) and PRLF are assumed. After that, the estimated values are calculated using the previous values. The estimates of DWCRTE, DWCRRE, and DWCRHCE are then obtained after 2,000 burn-in samples are subtracted from the 10,000 samples that the posterior density produced.

Convergence of the MCMC estimates for the DWCRTE, DWCRRE, and the DWCRHCE using M-H algorithms is shown in two figures in the case of complete sampling. Each figure shows the plot, histogram and cumulative mean where $t = 0.1$ and $c = 1.5$ in Figure 2 while Figure 3 represents the case where $t = 0.2$ and $c = 2.5$.

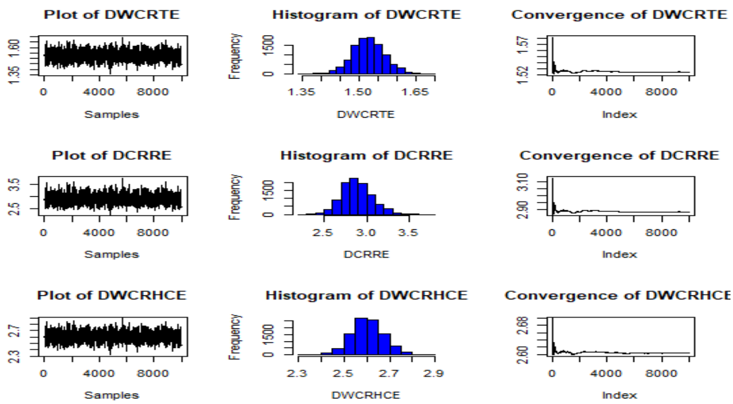

Figure 2. MCMC convergence of DWCRTE, DWCRRE, and DWCRHCE estimates at $t = 0.1$ and $c = 1.5$
Source: created by researchers utilizing the R programming language.

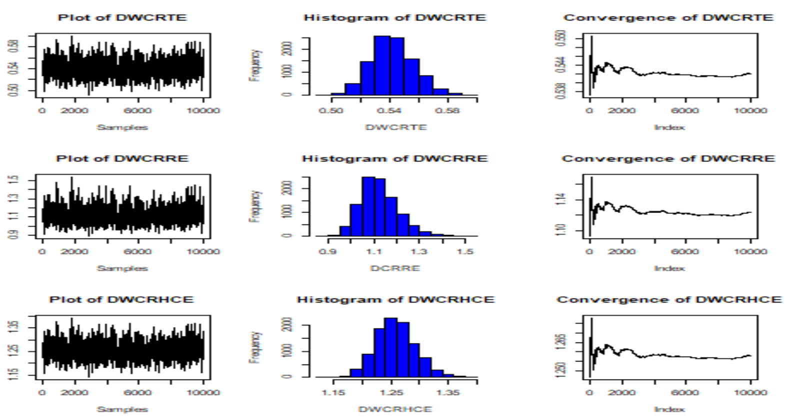

Figure 3. MCMC convergence of DWCRTE, DWCRRE, and DWCRHCE estimates at $t = 0.2$ and $c=2.5$
Source: created by researchers utilizing the R programming language.

Table 1. Estimates of different entropy measures under PT-IIC schemes given real dataset where m=15

Sch.	Method	$t = 0.1, c = 1.5$			$t = 0.2, c = 2.5$		
		DWCRTE	DWCRRE	DWCRHCE	DWCRTE	DWCRRE	DWCRHCE
S_1	MLE	1.5100	2.8129	2.5777	0.5322	1.0673	1.2349
	SLF	1.5249	2.8747	2.6032	0.5406	1.1102	1.2544
	LLF-1	1.5147	2.8321	2.5857	0.5401	1.1074	1.2531
	LLF-2	1.5612	3.0335	2.6651	0.5488	1.1550	1.2733
	PRLF	1.5309	2.9001	2.6134	0.5417	1.1159	1.2568
S_2	MLE	1.4986	2.7671	2.5583	0.5427	1.1217	1.2594
	SLF	1.4860	2.7175	2.5368	0.5414	1.1147	1.2563
	LLF-1	1.4828	2.7049	2.5313	0.5417	1.1162	1.2569
	LLF-2	1.4915	2.7388	2.5461	0.5415	1.1151	1.2565
	PRLF	1.4939	2.7484	2.5503	0.5431	1.1235	1.2601
S_3	MLE	1.4282	2.5043	2.4381	0.5105	0.9674	1.1845
	SLF	1.4396	2.5446	2.4576	0.5171	0.9964	1.1999
	LLF-1	1.4350	2.5281	2.4497	0.5185	1.0025	1.2031
	LLF-2	1.5249	2.8747	2.6031	0.5375	1.0939	1.2471
	PRLF	1.4561	2.6043	2.4858	0.5209	1.0136	1.2087
S_4	MLE	1.3820	2.3487	2.3591	0.4951	0.9047	1.1487
	SLF	1.3866	2.3638	2.3671	0.4998	0.9234	1.1597
	LLF-1	1.3971	2.3981	2.3849	0.5059	0.9484	1.1740
	LLF-2	1.3761	2.3298	2.3491	0.4932	0.8976	1.1445
	PRLF	1.3922	2.3822	2.3767	0.5006	0.9267	1.1616
complete	MLE	1.5228	2.8660	2.5996	0.5384	1.0986	1.2492
	SLF	1.5324	2.9066	2.6160	0.5434	1.1256	1.2610
	LLF-1	1.5272	2.8844	2.6071	0.5436	1.1265	1.2614
	LLF-2	1.5560	3.0104	2.6563	0.5485	1.1536	1.2728
	PRLF	1.5354	2.9195	2.6211	0.5438	1.1277	1.2619

Source: created by researchers utilizing the R programming language.

7. Simulation Study

This section evaluates the performance of estimate methodologies for BXIID under the PT-IIC scheme using a Monte Carlo simulation analysis. Specifically, maximum likelihood and Bayesian processes employing MCMC are considered. From the BXIID, we generate 1,000 random samples using the following guidelines:

1. Two cases of parameters of BXIID are assumed, namely: $(\delta, \lambda) =$ (1.5, 2.5) and(δ, λ) = (2.5, 1.5)
2. Two cases of parameters of weighted entropy measures are assumed, namely: (t, c) = (0.5, 1.5) and (t, c) = (1.5, 2.5).
3. The true values of different entropy measures are listed in Table 2.

Table 2. True values of different entropy measures

δ	λ	t	c	$T^*(c)$	$R^*(c)$	$H^*(c)$
1.5	2.5	0.5	1.5	1.3621	2.2853	2.3252
		1.5	2.5	0.3434	0.4825	0.7968
2.5	1.5	0.5	1.5	1.2567	1.9796	2.1453
		1.5	2.5	0.3943	0.5964	0.9115

Source: created by researchers utilizing the R programming language.

4. The sample size is assumed to be n = 40 and n = 60.
5. The number of stages in the PT-IIC scheme is m = 20, 30 at n = 40, m = 40 and 50 at n = 60.
6. Removed items r_i are assumed to have n and m values as shown in Table 3, where$r_m = n - m - \sum_{i=1}^{m-1} r_i$ and $r_i$is the number of failure items.

Based on the produced data and the previously-made assumptions, the MLEs are calculated using PT-IIC. After that, the MLEs are used to calculate $T^*(c)$, $R^*(c)$ and $H^*(c)$ given the values of t and c. For the Bayesian method, BEs using the M-H algorithm under different loss functions in the case of gamma prior are computed, where the following hyper-parameters are assumed: $(a_1, b_1, a_2, b_2) = (1.5, 2.5, 1.75, 2.75)$.

Different loss functions, including SLF, LLF-1($\tau = 0.5$), LLF-2($\tau = -0.5$) and PRLF, are given. These values are then employed to determine the estimated values. The average (Avg.) of all entropy estimates and the Avg. of the MSE are presented in Tables 4 (a) to 5 (d), which incorporate all of the Monte Carlo simulation's inputs.

Table 3. Numerous patterns for removing items from life test at different number of stages

(n, m)	Censoring Schemes			
	S_1	S_2	S_3	S_4
(40,20)	$(20, 0^{*19})$	$(10, 0^{*18}, 10)$	(1^{*20})	$(0^{*19}, 20)$
(40,30)	$(10, 0^{*29})$	$(5, 0^{*28}, 5)$	$(0^{*10}, 1^{*10}, 0^{*10})$	$(0^{*29}, 10)$
(60,40)	$(20, 0^{*39})$	$(10, 0^{*38}, 10)$	$(0^{*10}, 1^{*20}, 0^{*10})$	$(0^{*39}, 20)$
(60,50)	$(10, 0^{*49})$	$(5, 0^{*48}, 5)$	$(0^{*20}, 1^{*10}, 0^{*20})$	$(0^{*49}, 10)$

Here, $(2^{*4}, 0)$, for example, means that the censoring scheme employed is (2,2,2,2,0).

Source: created by researchers.

Table 4 (a). The Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at(δ, λ) = (1.5, 2.5), and (n, m) = (40, 20)

(*t*, *c*)	**Sch.**	**Estimate**	**DWCRTE**		**DWCRRE**		**DWCRHCE**	
			Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(0.5,1.5)	S_1	**MLE**	1.4108	0.0794	2.7101	2.6286	2.4084	0.2314
		SLF	0.9446	0.4437	1.4681	1.3845	1.6126	1.2929
		LLF-1	0.8564	0.5946	1.3158	1.6815	1.4620	1.7328
		LLF-2	1.0257	0.3293	1.6214	1.1383	1.7510	0.9596

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
		PRLF	1.0293	0.3130	1.6195	1.1060	1.7571	0.9121
	S_2	MLE	1.3937	0.2089	3.0689	12.2305	2.3792	0.6087
		SLF	0.4167	3.2758	0.8684	3.3052	0.7114	9.5464
		LLF-1	0.2216	4.7021	0.6644	3.9941	0.3783	13.7029
		LLF-2	0.5888	2.2816	1.0751	2.7086	1.0051	6.6492
		PRLF	0.6137	2.0044	1.0844	2.6074	1.0477	5.8414
	S_3	MLE	1.4055	0.1730	3.0630	10.8346	2.3994	0.5042
		SLF	0.4537	2.7111	0.8827	3.1863	0.7746	7.9007
		LLF-1	0.2675	3.8704	0.6797	3.8568	0.4566	11.2793
		LLF-2	0.6184	1.9003	1.0883	2.6073	1.0557	5.5378
		PRLF	0.6402	1.6882	1.0959	2.5141	1.0929	4.9198
	S_4	MLE	1.4494	0.1353	3.6653	12.6546	2.4743	0.3944
		SLF	0.7864	0.4795	1.1076	1.8434	1.3424	1.3972
		LLF-1	0.7149	0.5654	0.9811	2.1068	1.2205	1.6476
		LLF-2	0.8485	0.4207	1.2309	1.6438	1.4485	1.2260
		PRLF	0.8377	0.4300	1.2074	1.6717	1.4300	1.2532
(1.5,2.5)	S_1	MLE	0.3752	0.0147	0.6654	0.6670	0.8707	0.0792
		SLF	0.2018	0.0438	0.2752	0.0897	0.4682	0.2358
		LLF-1	0.1756	0.0545	0.2384	0.1057	0.4076	0.2936
		LLF-2	0.2271	0.0349	0.3130	0.0763	0.5269	0.1880
		PRLF	0.2281	0.0337	0.3130	0.0742	0.5292	0.1816
	S_2	MLE	0.3793	0.0274	0.8829	2.5356	0.8800	0.1473
		SLF	0.0796	0.1314	0.1351	0.1853	0.1846	0.7077
		LLF-1	0.0394	0.1632	0.0905	0.2174	0.0913	0.8786
		LLF-2	0.1184	0.1049	0.1815	0.1565	0.2747	0.5647
		PRLF	0.1223	0.0994	0.1836	0.1518	0.2839	0.5350
	S_3	MLE	0.3818	0.0250	0.8531	2.2068	0.8859	0.1347
		SLF	0.0848	0.1228	0.1380	0.1797	0.1968	0.6611
		LLF-1	0.0453	0.1526	0.0937	0.2113	0.1051	0.8215
		LLF-2	0.1230	0.0979	0.1843	0.1516	0.2855	0.5271
		PRLF	0.1265	0.0930	0.1860	0.1471	0.2936	0.5008
	S_4	MLE	0.4105	0.0254	1.1969	4.9892	0.9526	0.1366
		SLF	0.1754	0.0390	0.2198	0.0918	0.4071	0.2098
		LLF-1	0.1592	0.0436	0.1955	0.1015	0.3694	0.2348
		LLF-2	0.1901	0.0358	0.2436	0.0849	0.4412	0.1926
		PRLF	0.1878	0.0360	0.2394	0.0855	0.4358	0.1941

Source: created by researchers utilizing the R programming language.

Table 4 (b). Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at $(\delta, \lambda) = (1.5, 2.5)$ and $(n, m) = (40, 30)$

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.4217	0.0503	2.6143	0.6428	2.4270	0.1465
		SLF	1.1547	0.1414	1.8470	0.6809	1.9712	0.4122
		LLF-1	1.1107	0.1751	1.7480	0.7857	1.8961	0.5102
		LLF-2	1.1966	0.1143	1.9466	0.5968	2.0427	0.3330
		PRLF	1.1964	0.1113	1.9420	0.5834	2.0424	0.3243
	S_2	MLE	1.4242	0.0626	2.6550	0.7982	2.4313	0.1825
		SLF	1.0862	0.2244	1.7193	0.9094	1.8543	0.6540
		LLF-1	1.0335	0.2789	1.6101	1.0545	1.7643	0.8128
		LLF-2	1.1362	0.1802	1.8293	0.7898	1.9397	0.5252
		PRLF	1.1370	0.1744	1.8256	0.7714	1.9411	0.5082
	S_3	MLE	1.4242	0.0601	2.6487	0.7695	2.4313	0.1752
		SLF	1.0926	0.2134	1.7294	0.8854	1.8653	0.6219
		LLF-1	1.0415	0.2643	1.6224	1.0243	1.7779	0.7703
		LLF-2	1.1412	0.1719	1.8373	0.7709	1.9482	0.5011
		PRLF	1.1416	0.1668	1.8327	0.7538	1.9488	0.4861
	S_4	MLE	1.4242	0.0785	2.7197	1.5945	2.4313	0.2287
		SLF	1.0046	0.3581	1.5816	1.1984	1.7149	1.0435
		LLF-1	0.9404	0.4489	1.4612	1.3958	1.6054	1.3081
		LLF-2	1.0651	0.2848	1.7036	1.0321	1.8182	0.8300
		PRLF	1.0673	0.2734	1.7008	1.0069	1.8221	0.7967
(1.5,2.5)	S_1	MLE	0.3770	0.0103	0.6030	0.1413	0.8747	0.0554
		SLF	0.2679	0.0189	0.3700	0.0496	0.6217	0.1016
		LLF-1	0.2525	0.0224	0.3444	0.0556	0.5859	0.1204
		LLF-2	0.2830	0.0160	0.3960	0.0449	0.6567	0.0859
		PRLF	0.2831	0.0156	0.3954	0.0439	0.6570	0.0839
	S_2	MLE	0.3799	0.0127	0.6378	0.3206	0.8815	0.0685
		SLF	0.2445	0.0271	0.3361	0.0637	0.5674	0.1457
		LLF-1	0.2270	0.0321	0.3085	0.0719	0.5266	0.1728
		LLF-2	0.2617	0.0228	0.3643	0.0571	0.6072	0.1227
		PRLF	0.2621	0.0222	0.3639	0.0558	0.6081	0.1195
	S_3	MLE	0.3792	0.0121	0.6334	0.3127	0.8799	0.0651
		SLF	0.2468	0.0259	0.3388	0.0619	0.5726	0.1392
		LLF-1	0.2297	0.0306	0.3118	0.0697	0.5330	0.1649
		LLF-2	0.2635	0.0218	0.3664	0.0555	0.6114	0.1175
		PRLF	0.2637	0.0213	0.3657	0.0543	0.6118	0.1147
	S_4	MLE	0.3821	0.0155	0.7030	0.7839	0.8865	0.0836
		SLF	0.2189	0.0380	0.3008	0.0804	0.5079	0.2043
		LLF-1	0.1988	0.0451	0.2711	0.0913	0.4614	0.2429
		LLF-2	0.2385	0.0318	0.3314	0.0713	0.5534	0.1714
		PRLF	0.2392	0.0309	0.3312	0.0696	0.5550	0.1664

Source: created by researchers utilizing the R programming language.

Table 4(c). The Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at $(n, (\delta, \lambda) = (2.5, 1.5)$ and $(n, m) = (60, 40)$

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.4210	0.0362	2.5731	0.4596	2.4258	0.1055
		SLF	1.2338	0.0707	2.0056	0.4220	2.1062	0.2060
		LLF-1	1.2049	0.0839	1.9326	0.4710	2.0569	0.2445
		LLF-2	1.2617	0.0598	2.0792	0.3841	2.1539	0.1744
		PRLF	1.2606	0.0589	2.0740	0.3773	2.1520	0.1715
	S_2	MLE	1.4226	0.0486	2.6113	0.6132	2.4285	0.1415
		SLF	1.1639	0.1287	1.8595	0.6310	1.9870	0.3751
		LLF-1	1.1274	0.1540	1.7750	0.7132	1.9246	0.4487
		LLF-2	1.1992	0.1076	1.9450	0.5639	2.0472	0.3137
		PRLF	1.1986	0.1054	1.9400	0.5537	2.0461	0.3072
	S_3	MLE	1.4234	0.0457	2.6074	0.5854	2.4300	0.1331
		SLF	1.1741	0.1171	1.8784	0.5962	2.0043	0.3413
		LLF-1	1.1392	0.1396	1.7965	0.6718	1.9448	0.4069
		LLF-2	1.2078	0.0983	1.9612	0.5346	2.0619	0.2865
		PRLF	1.2068	0.0966	1.9554	0.5257	2.0601	0.2814
	S_4	MLE	1.4209	0.0642	2.6426	0.7779	2.4256	0.1871
		SLF	1.0802	0.2275	1.7009	0.9062	1.8440	0.6629
		LLF-1	1.0334	0.2748	1.6034	1.0341	1.7642	0.8007
		LLF-2	1.1251	0.1879	1.8001	0.7982	1.9207	0.5476
		PRLF	1.1253	0.1829	1.7953	0.7834	1.9209	0.5329
(1.5,2.5)	S_1	MLE	0.3734	0.0073	0.5727	0.0428	0.8663	0.0395
		SLF	0.2944	0.0107	0.4083	0.0323	0.6830	0.0576
		LLF-1	0.2836	0.0123	0.3890	0.0353	0.6579	0.0662
		LLF-2	0.3050	0.0094	0.4279	0.0301	0.7078	0.0505
		PRLF	0.3048	0.0092	0.4270	0.0296	0.7072	0.0497
	S_2	MLE	0.3757	0.0098	0.5874	0.0588	0.8717	0.0525
		SLF	0.2686	0.0176	0.3687	0.0463	0.6234	0.0948
		LLF-1	0.2557	0.0204	0.3469	0.0512	0.5932	0.1096
		LLF-2	0.2815	0.0153	0.3909	0.0423	0.6531	0.0822
		PRLF	0.2814	0.0150	0.3901	0.0415	0.6529	0.0807
	S_3	MLE	0.3754	0.0091	0.5843	0.0550	0.8710	0.0489
		SLF	0.2725	0.0161	0.3740	0.0436	0.6323	0.0869
		LLF-1	0.2601	0.0186	0.3529	0.0481	0.6035	0.1002
		LLF-2	0.2848	0.0140	0.3954	0.0400	0.6608	0.0755
		PRLF	0.2845	0.0138	0.3944	0.0393	0.6602	0.0743
	S_4	MLE	0.3771	0.0125	0.6227	0.2506	0.8749	0.0673
		SLF	0.2405	0.0272	0.3275	0.0631	0.5580	0.1466
		LLF-1	0.2249	0.0316	0.3031	0.0705	0.5219	0.1704
		LLF-2	0.2558	0.0234	0.3527	0.0568	0.5937	0.1260
		PRLF	0.2559	0.0229	0.3519	0.0558	0.5939	0.1234

Source: created by researchers utilizing the R programming language.

Table 4(d). The Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at $(\delta, \lambda) = (2.5, 1.5)$ and (n, m) = (60, 50)

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.4218	0.0295	2.5565	0.3722	2.4272	0.0861
		SLF	1.2762	0.0462	2.1027	0.3103	2.1787	0.1347
		LLF-1	1.2546	0.0531	2.0441	0.3366	2.1417	0.1549
		LLF-2	1.2973	0.0405	2.1615	0.2912	2.2147	0.1180
		PRLF	1.2962	0.0400	2.1569	0.2864	2.2128	0.1165
	S_2	MLE	1.4231	0.0332	2.5708	0.4189	2.4295	0.0966
		SLF	1.2558	0.0580	2.0556	0.3618	2.1439	0.1690
		LLF-1	1.2323	0.0670	1.9938	0.3952	2.1037	0.1953
		LLF-2	1.2787	0.0505	2.1177	0.3365	2.1829	0.1472
		PRLF	1.2777	0.0498	2.1131	0.3310	2.1812	0.1451
	S_3	MLE	1.4231	0.0326	2.5692	0.4127	2.4293	0.0951
		SLF	1.2570	0.0570	2.0581	0.3586	2.1459	0.1661
		LLF-1	1.2339	0.0657	1.9972	0.3912	2.1064	0.1916
		LLF-2	1.2796	0.0497	2.1194	0.3338	2.1844	0.1449
		PRLF	1.2784	0.0491	2.1145	0.3285	2.1824	0.1429
	S_4	MLE	1.4237	0.0373	2.5833	0.4680	2.4303	0.1087
		SLF	1.2340	0.0726	2.0070	0.4215	2.1066	0.2116
		LLF-1	1.2083	0.0843	1.9414	0.4635	2.0627	0.2457
		LLF-2	1.2591	0.0628	2.0732	0.3886	2.1494	0.1830
		PRLF	1.2581	0.0618	2.0686	0.3822	2.1477	0.1802
(1.5,2.5)	S_1	MLE	0.3726	0.0060	0.5657	0.0341	0.8646	0.0323
		SLF	0.3098	0.0074	0.4331	0.0246	0.7189	0.0401
		LLF-1	0.3014	0.0083	0.4174	0.0262	0.6993	0.0449
		LLF-2	0.3181	0.0067	0.4490	0.0235	0.7382	0.0362
		PRLF	0.3179	0.0066	0.4482	0.0231	0.7377	0.0357
	S_2	MLE	0.3737	0.0068	0.5710	0.0391	0.8671	0.0364
		SLF	0.3017	0.0091	0.4198	0.0284	0.7001	0.0489
		LLF-1	0.2927	0.0102	0.4034	0.0304	0.6792	0.0549
		LLF-2	0.3106	0.0081	0.4365	0.0269	0.7208	0.0438
		PRLF	0.3104	0.0080	0.4358	0.0265	0.7203	0.0432
	S_3	MLE	0.3734	0.0066	0.5698	0.0380	0.8665	0.0355
		SLF	0.3023	0.0089	0.4207	0.0279	0.7015	0.0478
		LLF-1	0.2935	0.0100	0.4045	0.0299	0.6810	0.0536
		LLF-2	0.3110	0.0080	0.4370	0.0265	0.7217	0.0429
		PRLF	0.3108	0.0079	0.4362	0.0260	0.7211	0.0423
	S_4	MLE	0.3744	0.0076	0.5757	0.0441	0.8688	0.0407
		SLF	0.2934	0.0110	0.4065	0.0325	0.6807	0.0591
		LLF-1	0.2837	0.0124	0.3892	0.0351	0.6582	0.0667
		LLF-2	0.3029	0.0098	0.4241	0.0306	0.7029	0.0526
		PRLF	0.3028	0.0096	0.4234	0.0301	0.7025	0.0518

Source: created by researchers utilizing the R programming language.

Table 5(a). The Avg. and MSE, of different weighted entropy estimates for BXIID under PT-IIC schemes at $(\delta, \lambda) = (1.5, 2.5)$ and $(n, m) = (40, 20)$

(*t*, *c*)	**Sch.**	**Estimate**	**DWCRTE**		**DWCRRE**		**DWCRHCE**	
			Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(0.5,1.5)	S_1	**MLE**	1.3067	0.0516	2.2314	0.7609	2.2307	0.1504
		SLF	1.0486	0.1430	1.5868	0.5515	1.7901	0.4168
		LLF-1	0.9962	0.1856	1.4846	0.6601	1.7007	0.5410
		LLF-2	1.0976	0.1101	1.6882	0.4660	1.8738	0.3209
		PRLF	1.1056	0.1024	1.7018	0.4431	1.8873	0.2983
	S_2	**MLE**	1.3135	0.0780	2.3729	3.1072	2.2422	0.2274
		SLF	0.8639	0.4090	1.2871	1.0673	1.4748	1.1918
		LLF-1	0.7868	0.5359	1.1649	1.2812	1.3432	1.5618
		LLF-2	0.9345	0.3116	1.4074	0.8892	1.5953	0.9080
		PRLF	0.9491	0.2842	1.4277	0.8407	1.6202	0.8282
	S_3	**MLE**	1.3174	0.0793	2.4249	4.2087	2.2489	0.2311
		SLF	0.8543	0.4305	1.2749	1.1017	1.4584	1.2547
		LLF-1	0.7775	0.5607	1.1538	1.3151	1.3272	1.6339
		LLF-2	0.9247	0.3304	1.3942	0.9238	1.5786	0.9628
		PRLF	0.9402	0.3007	1.4157	0.8724	1.6050	0.8763
	S_4	**MLE**	1.3107	0.1344	2.5900	9.0307	2.2375	0.3917
		SLF	0.5819	1.4653	0.9524	1.9576	0.9934	4.2703
		LLF-1	0.4553	2.0057	0.8024	2.3463	0.7773	5.8452
		LLF-2	0.6946	1.0729	1.0997	1.6268	1.1857	3.1266
		PRLF	0.7249	0.9285	1.1309	1.5257	1.2374	2.7059
(1.5,2.5)	S_1	**MLE**	0.4189	0.0078	0.7511	0.5409	0.9721	0.0420
		SLF	0.3141	0.0171	0.4534	0.0597	0.7288	0.0921
		LLF-1	0.2966	0.0213	0.4210	0.0698	0.6883	0.1145
		LLF-2	0.3310	0.0137	0.4864	0.0517	0.7681	0.0738
		PRLF	0.3328	0.0130	0.4889	0.0497	0.7722	0.0702
	S_2	**MLE**	0.4250	0.0121	0.9170	1.9017	0.9862	0.0654
		SLF	0.2601	0.0374	0.3672	0.1030	0.6036	0.2012
		LLF-1	0.2378	0.0459	0.3312	0.1202	0.5518	0.2473
		LLF-2	0.2816	0.0302	0.4039	0.0883	0.6534	0.1628
		PRLF	0.2847	0.0286	0.4079	0.0849	0.6607	0.1539
	S_3	**MLE**	0.4259	0.0120	0.9255	2.0453	0.9882	0.0648
		SLF	0.2591	0.0379	0.3661	0.1043	0.6013	0.2041
		LLF-1	0.2371	0.0464	0.3305	0.1212	0.5502	0.2496
		LLF-2	0.2803	0.0308	0.4022	0.0899	0.6503	0.1661
		PRLF	0.2836	0.0291	0.4065	0.0863	0.6581	0.1568
	S_4	**MLE**	0.4289	0.0179	1.1547	4.2872	0.9952	0.0965
		SLF	0.1958	0.0752	0.2797	0.1632	0.4543	0.4047
		LLF-1	0.1666	0.0921	0.2391	0.1898	0.3865	0.4958
		LLF-2	0.2237	0.0610	0.3211	0.1398	0.5190	0.3283
		PRLF	0.2291	0.0569	0.3272	0.1338	0.5316	0.3066

Source: created by researchers utilizing the R programming language.

Table 5(b). The Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at $(\delta, \lambda) = (1.5, 2.5)$ and (n, m) = (40, 30)

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.3110	0.0342	2.1955	0.3041	2.2381	0.0995
		SLF	1.1581	0.0580	1.7958	0.2938	1.9770	0.1692
		LLF-1	1.1276	0.0706	1.7270	0.3326	1.9250	0.2057
		LLF-2	1.1872	0.0482	1.8640	0.2651	2.0267	0.1405
		PRLF	1.1907	0.0461	1.8710	0.2569	2.0327	0.1344
	S_2	MLE	1.3152	0.0380	2.2154	0.3439	2.2453	0.1108
		SLF	1.1294	0.0749	1.7371	0.3508	1.9281	0.2183
		LLF-1	1.0968	0.0911	1.6660	0.3996	1.8724	0.2656
		LLF-2	1.1605	0.0619	1.8075	0.3128	1.9811	0.1805
		PRLF	1.1645	0.0592	1.8150	0.3028	1.9879	0.1724
	S_3	MLE	1.3155	0.0391	2.2181	0.3516	2.2457	0.1140
		SLF	1.1232	0.0810	1.7264	0.3695	1.9174	0.2359
		LLF-1	1.0909	0.0979	1.6567	0.4192	1.8623	0.2853
		LLF-2	1.1539	0.0673	1.7955	0.3305	1.9699	0.1961
		PRLF	1.1581	0.0643	1.8034	0.3199	1.9770	0.1873
	S_4	MLE	1.3186	0.0432	2.2352	0.3925	2.2510	0.1258
		SLF	1.0939	0.1014	1.6684	0.4316	1.8674	0.2954
		LLF-1	1.0580	0.1235	1.5936	0.4944	1.8061	0.3600
		LLF-2	1.1280	0.0834	1.7424	0.3812	1.9256	0.2431
		PRLF	1.1326	0.0795	1.7508	0.3684	1.9335	0.2317
(1.5,2.5)	S_1	MLE	0.4198	0.0054	0.6888	0.0451	0.9740	0.0292
		SLF	0.3519	0.0082	0.5220	0.0349	0.8166	0.0439
		LLF-1	0.3406	0.0097	0.4983	0.0389	0.7903	0.0522
		LLF-2	0.3630	0.0069	0.5460	0.0321	0.8424	0.0373
		PRLF	0.3637	0.0067	0.5469	0.0312	0.8439	0.0361
	S_2	MLE	0.4219	0.0062	0.7046	0.0948	0.9791	0.0335
		SLF	0.3421	0.0102	0.5037	0.0411	0.7937	0.0551
		LLF-1	0.3301	0.0121	0.4795	0.0460	0.7660	0.0653
		LLF-2	0.3537	0.0087	0.5282	0.0375	0.8207	0.0467
		PRLF	0.3545	0.0084	0.5293	0.0365	0.8225	0.0452
	S_3	MLE	0.4217	0.0063	0.7094	0.1337	0.9784	0.0341
		SLF	0.3406	0.0107	0.5014	0.0425	0.7903	0.0577
		LLF-1	0.3289	0.0126	0.4778	0.0474	0.7632	0.0681
		LLF-2	0.3520	0.0091	0.5252	0.0388	0.8167	0.0491
		PRLF	0.3528	0.0088	0.5265	0.0378	0.8187	0.0476
	S_4	MLE	0.4239	0.0073	0.7324	0.2407	0.9837	0.0390
		SLF	0.3301	0.0132	0.4824	0.0493	0.7660	0.0709
		LLF-1	0.3173	0.0156	0.4573	0.0553	0.7363	0.0841
		LLF-2	0.3426	0.0111	0.5078	0.0445	0.7949	0.0600
		PRLF	0.3435	0.0108	0.5091	0.0434	0.7971	0.0580

Source: created by researchers utilizing the R programming language.

Table 5(c). The Avg. and MSE, of different weighted entropy estimates for BXIID under PT-IIC schemes at $(\delta, \lambda) = (2.5,1.5)$ and (n, m) = (60,40)

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.3122	0.0258	2.1817	0.2285	2.2401	0.0751
		SLF	1.2038	0.0334	1.8889	0.1946	2.0550	0.0974
		LLF-1	1.1833	0.0385	1.8390	0.2104	2.0199	0.1121
		LLF-2	1.2237	0.0294	1.9385	0.1839	2.0891	0.0858
		PRLF	1.2259	0.0286	1.9432	0.1800	2.0928	0.0832
	S_2	MLE	1.3162	0.0300	2.2017	0.2704	2.2469	0.0874
		SLF	1.1725	0.0464	1.8201	0.2450	2.0016	0.1353
		LLF-1	1.1502	0.0538	1.7681	0.2688	1.9635	0.1569
		LLF-2	1.1941	0.0404	1.8718	0.2268	2.0384	0.1176
		PRLF	1.1966	0.0391	1.8771	0.2215	2.0427	0.1138
	S_3	MLE	1.3166	0.0308	2.2042	0.2768	2.2476	0.0896
		SLF	1.1687	0.0494	1.8136	0.2570	1.9951	0.1440
		LLF-1	1.1469	0.0570	1.7631	0.2808	1.9580	0.1660
		LLF-2	1.1898	0.0432	1.8638	0.2387	2.0311	0.1259
		PRLF	1.1925	0.0418	1.8695	0.2331	2.0358	0.1218
	S_4	MLE	1.3190	0.0358	2.2210	0.3236	2.2516	0.1042
		SLF	1.1347	0.0678	1.7424	0.3206	1.9370	0.1977
		LLF-1	1.1093	0.0792	1.6861	0.3558	1.8937	0.2308
		LLF-2	1.1591	0.0584	1.7983	0.2923	1.9787	0.1701
		PRLF	1.1622	0.0563	1.8045	0.2848	1.9841	0.1641
(1.5,2.5)	S_1	MLE	0.4174	0.0039	0.6745	0.0313	0.9686	0.0213
		SLF	0.3685	0.0048	0.5522	0.0233	0.8551	0.0261
		LLF-1	0.3605	0.0055	0.5345	0.0251	0.8365	0.0298
		LLF-2	0.3764	0.0043	0.5701	0.0221	0.8733	0.0231
		PRLF	0.3767	0.0042	0.5706	0.0217	0.8742	0.0226
	S_2	MLE	0.4195	0.0048	0.6845	0.0393	0.9734	0.0258
		SLF	0.3573	0.0066	0.5304	0.0294	0.8291	0.0358
		LLF-1	0.3488	0.0076	0.5122	0.0320	0.8094	0.0409
		LLF-2	0.3657	0.0058	0.5488	0.0275	0.8485	0.0315
		PRLF	0.3662	0.0057	0.5495	0.0270	0.8496	0.0307
	S_3	MLE	0.4192	0.0048	0.6839	0.0393	0.9728	0.0259
		SLF	0.3568	0.0069	0.5297	0.0302	0.8279	0.0369
		LLF-1	0.3486	0.0078	0.5122	0.0327	0.8089	0.0420
		LLF-2	0.3648	0.0061	0.5474	0.0284	0.8465	0.0327
		PRLF	0.3654	0.0059	0.5483	0.0278	0.8478	0.0319
	S_4	MLE	0.4211	0.0058	0.6993	0.0829	0.9772	0.0313
		SLF	0.3442	0.0093	0.5060	0.0376	0.7988	0.0499
		LLF-1	0.3349	0.0106	0.4866	0.0411	0.7770	0.0573
		LLF-2	0.3534	0.0081	0.5255	0.0347	0.8201	0.0437
		PRLF	0.3541	0.0079	0.5264	0.0340	0.8216	0.0426

Source: created by researchers utilizing the R programming language.

Table 5(d). The Avg. and MSE of different weighted entropy estimates for BXIID under PT-IIC schemes at$(\delta, \lambda) = (2.5{,}1.5)$ and $(n, m) = (60, 50)$

(t, c)	Sch.	Estimate	DWCRTE		DWCRRE		DWCRHCE	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(0.5,1.5)	S_1	MLE	1.3126	0.0213	2.1735	0.1882	2.2407	0.0620
		SLF	1.2278	0.0238	1.9409	0.1514	2.0960	0.0694
		LLF-1	1.2117	0.0265	1.9003	0.1590	2.0685	0.0773
		LLF-2	1.2435	0.0218	1.9812	0.1470	2.1228	0.0635
		PRLF	1.2450	0.0213	1.9846	0.1448	2.1254	0.0621
	S_2	MLE	1.3142	0.0224	2.1803	0.1996	2.2434	0.0653
		SLF	1.2190	0.0263	1.9201	0.1614	2.0809	0.0766
		LLF-1	1.2026	0.0294	1.8794	0.1707	2.0530	0.0856
		LLF-2	1.2349	0.0239	1.9606	0.1555	2.1081	0.0696
		PRLF	1.2365	0.0233	1.9641	0.1530	2.1108	0.0680
	S_3	MLE	1.3145	0.0229	2.1820	0.2035	2.2440	0.0666
		SLF	1.2161	0.0276	1.9141	0.1672	2.0760	0.0803
		LLF-1	1.1999	0.0308	1.8740	0.1769	2.0484	0.0897
		LLF-2	1.2319	0.0250	1.9540	0.1608	2.1029	0.0729
		PRLF	1.2335	0.0245	1.9577	0.1582	2.1057	0.0713
	S_4	MLE	1.3156	0.0238	2.1871	0.2132	2.2458	0.0695
		SLF	1.2088	0.0297	1.8970	0.1757	2.0635	0.0866
		LLF-1	1.1919	0.0334	1.8555	0.1872	2.0347	0.0973
		LLF-2	1.2252	0.0268	1.9383	0.1678	2.0916	0.0781
		PRLF	1.2269	0.0262	1.9419	0.1649	2.0944	0.0762
(1.5,2.5)	S_1	MLE	0.4170	0.0032	0.6694	0.0252	0.9675	0.0174
		SLF	0.3775	0.0035	0.5698	0.0181	0.8760	0.0188
		LLF-1	0.3711	0.0039	0.5551	0.0191	0.8611	0.0210
		LLF-2	0.3839	0.0032	0.5847	0.0176	0.8908	0.0172
		PRLF	0.3841	0.0031	0.5850	0.0173	0.8913	0.0169
	S_2	MLE	0.4177	0.0035	0.6726	0.0274	0.9693	0.0187
		SLF	0.3743	0.0039	0.5632	0.0196	0.8685	0.0209
		LLF-1	0.3678	0.0043	0.5485	0.0207	0.8534	0.0233
		LLF-2	0.3807	0.0035	0.5781	0.0189	0.8835	0.0190
		PRLF	0.3810	0.0035	0.5784	0.0186	0.8840	0.0187
	S_3	MLE	0.4177	0.0035	0.6727	0.0278	0.9692	0.0190
		SLF	0.3735	0.0040	0.5617	0.0201	0.8667	0.0217
		LLF-1	0.3671	0.0045	0.5473	0.0213	0.8518	0.0242
		LLF-2	0.3798	0.0037	0.5763	0.0194	0.8813	0.0197
		PRLF	0.3801	0.0036	0.5767	0.0191	0.8820	0.0194
	S_4	MLE	0.4184	0.0038	0.6759	0.0300	0.9709	0.0202
		SLF	0.3705	0.0044	0.5557	0.0215	0.8598	0.0237
		LLF-1	0.3639	0.0049	0.5407	0.0228	0.8443	0.0265
		LLF-2	0.3771	0.0040	0.5708	0.0206	0.8751	0.0215
		PRLF	0.3774	0.0039	0.5712	0.0203	0.8757	0.0211

Source: created by researchers utilizing the R programming language.

It is evident from the tabulated result values that:

1- As n and m increase, the MSE of the Bayes estimate gradually decreases.

2- In comparison to other estimates, the MSE of all estimates based on DWCRTE frequently yields the smallest values.

3- As can be seen in Tables 4 (a) to 4 (d), in the majority of the cases, the precision measures of DWCRTE, DWCRRE and DWCRHCE under PRLF and LLF-2 are preferable to the corresponding estimates under LLF-1 and SLF for all schemes.

4- At true values $T^*(c) = 0.3434$ and 1.3621, the BEs of $T^*(c)$ under all loss functions are preferred over the other entropy measures for all schemes (see Tables 4(a) and 4(d)).

5- The MLEs and BEs of $T^*(c)$, $R^*(c)$ and $H^*(c)$ under different loss functions are decreasing as n and m increase from (40, 20) to (60, 50) (see Tables 4(a) and 4(d)).

6- The MSE of all BEs of $T^*(c)$ gets the smallest values compared to the others for all schemes at $(n, m) = (60, 40)$ and $(t, c) = (0.5, 1.5)$ and $(1.5, 2.5)$ (see Table 5(c)).

7- In most cases, the MSEs of all estimates have the largest values in the case of S_3 and S_4 compared to other schemes, at $(n, m) = (60, 50)$, $(t, c) = (0.5, 1.5)$ and $(1.5, 2.5)$ (see Table 5(d)).

8- The MSE of $T^*(c)$ gets the smallest values under all loss functions in the case of S_1 at $m = 20$, $n = 40$ (see Figure 4).

9- At true values of $H^*(c) = 0.7968$ and $T^*(c) = 0.3434$, it can be observed that the BE of DWCRHCE gets the largest MSE, while the BE of DWCRTE gets the smallest value for all loss functions in Sch.1 (see Figure 5).

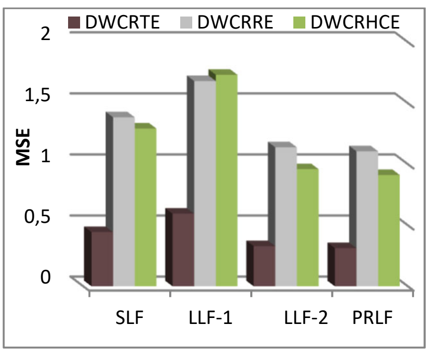

Figure 4. The MSE of different entropy estimates at $t = 0.5$

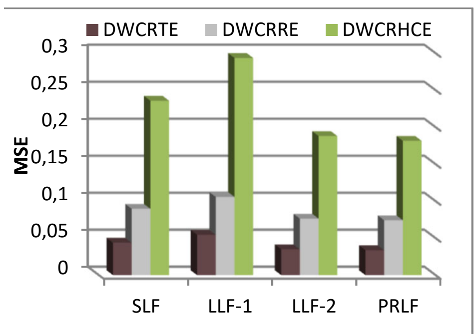

Figure 5. The MSE of different entropy estimates at $t = 1.5$

Source: created by researchers utilizing Microsoft Excel.

10- The MSE of the BE for $R^*(c)$ based on S_1 has the biggest values at m = 50, n = 60 and t =0.5 (see Figure 6).

11- The MSE of the DWCRTE estimate under different loss functions based on S_1 typically produces the smallest values when compared to other estimates at m = 50, n = 60 and t=1.5 (see Figure 7).

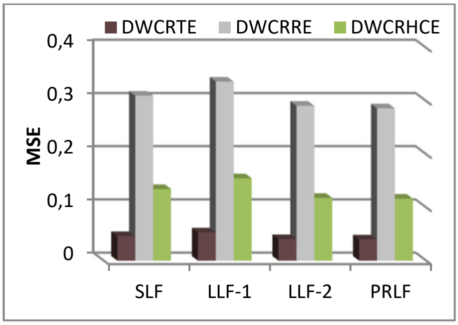

Figure 6. MSE of different Entropy estimates at t = 0.5, m = 50, and n = 60

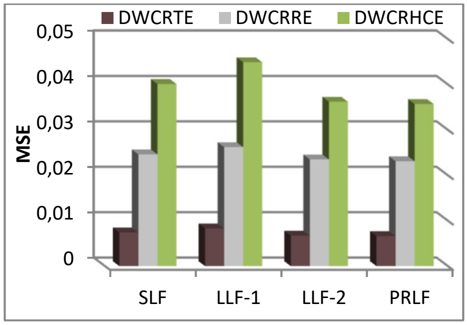

Figure 7. MSE of different Entropy estimates at t = 1.5

Source: created by researchers utilizing Microsoft Excel.

12- It can be observed from Figure 8 that the MSE of $\hat{T}^*(c)$ under PRLF takes the smallest values compared to the others in Sch. 4 at m = 20, n = 40, and t = 0.5.

13- At m = 20, n = 40, where the true value of $\hat{T}^*(c)$ = 0.3434, the MSE of $\hat{T}^*(c)$ under LLF-2 takes the smallest values compared to the others under S_4 (Figure 9).

14- When compared to the other estimates from S_2, S_3, and S_4, the MSE of all entropy estimates based on S_1 often has the smallest values. The majority of entropy estimates (MLE, SLF, LLF-1, LLF-2, and PRLF) show a slight decrease as t increases.

15- It should be highlighted that, in comparison to the other estimates based on S_4, S_3 often produces the shortest MSE outcomes.

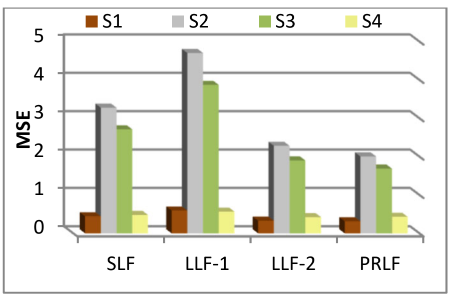

Figure 8. MSE of DWCRTE for different loss functions when t = 0.5

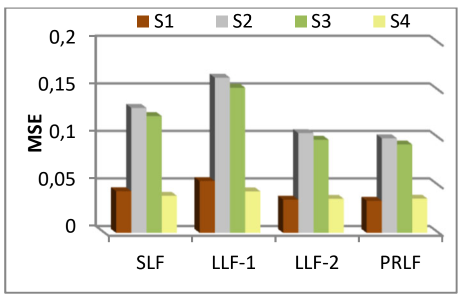

Figure 9. MSE of DWCRTE for different loss functions when t =1.5

Source: created by researchers utilizing Microsoft Excel.

16- As can be seen in Figures 10 and 11, as n and m increase, the MSEs of $R^*(c)$ under different loss functions are decreasing.

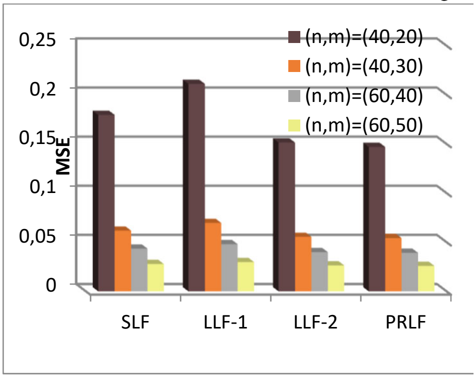

Figure 10. MSE of DWCRRE for different estimates under S_3 at $\delta = 1.5, \lambda = 2.5$ and t =1.5

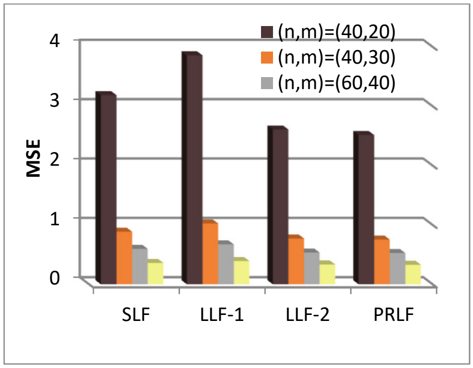

Figure 9. MSE of DWCRRE for different estimates under S_3 at$\delta = 1.5, \lambda = 2.5$ and t =0.5

Source: created by researchers utilizing Microsoft Excel.

8. Concluding Remarks

This article introduces the DWCRHC as an additional measure of uncertainty surrounding the residual lifetime function, particularly relevant in fields like survival analysis and reliability. The DWCRHC measure is formally defined within this work. We investigate the estimation of the DWCRHC, along with its related measures, DWCRRE and DWCRTE, for the BXIID under PT-IIC. Maximum and Bayesian estimation methods are employed. For the Bayesian estimation, we utilize the MCMC approach with the M-H algorithm, assuming a gamma prior distribution and considering three different loss functions. The article features the application, simulation studies, and an evaluation of the accuracy of the DWCRTE, DWCRRE, and DWCRHCE estimates for the BXIID.

Simulation results demonstrate that the BE of DWCRTE converges to the true value as the sample size increases. Generally, BEs under the PRLF exhibit the lowest MSE values, followed by the LLF-2, making them preferable over competing estimates. Furthermore, Sch. 1, compared to other schemes, often yields the lowest MSE values, followed in most cases by Sch. 2. The conclusions drawn from the simulated data are corroborated by the examination of actual data, particularly the water capacity data from the Shasta reservoir. These results are helpful in making well-informed decisions about the management of water resources. Future research could explore the application of the E-Bayesian technique to estimate other uncertainty metrics, such as dynamic weighted cumulative residual Shannon entropy.

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