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Advanced Censoring Schemes for Statistical Inference of Reliability in Engineering Contexts

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Abstract. In practice, systems that are exposed to rigorous operational environments frequently fail. The critical concept that system failure often occurs when these extremely strict operating constraints are reached has not yet been thoroughly investigated by researchers. This study address this gap in an analysis of the multi-stress-strength model $\Upsilon = P(U < W < V)$, in which stresses (U &V) and strength (W) are defined using the exponentiated Weibull distribution. In this work, point and interval estimators for Υ under a generalized progressive hybrid censoring scheme are derived. Using symmetric and asymmetric loss functions, we derive Bayesian estimators and maximum likelihood estimators. We employ Markov chain Monte Carlo techniques because the Bayesian estimators are computationally complex. Furthermore, to assess estimator performance, we create percentile bootstrap intervals, bootstrap-t intervals, and Bayesian credible intervals. A simulation study was conducted to evaluate the efficacy of the proposed estimates. Numerical results lead us to the conclusion that the Bayesian estimates based on informative priors outperform classical estimates in terms of biases and mean squared errors. The highest posterior density credible intervals offer a more favorable average length than asymptotic confidence intervals when performing Bayesian estimation. Real progressively censored engineering data applications of real data are presented to demonstrate the effectiveness of the proposed estimators.

2020 Mathematics Subject Classifications: 62N01, 62F10, 62F15, 62P30

Key Words and Phrases: Stress-strength model, Exponentiated Weibull, Generalized progressive hybrid censoring, Maximum likelihood method, Bayesian inference

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1. Introduction

A significant issue in statistical literature is the analysis of the stress-strength model (SSM). The probability that the stress X will exceed the strength Y is represented by the parameter R, defined as $R = Pr(X < Y)$. The SSM is commonly utilized in practical engineering for product life testing, material reliability analysis, and design. For instance, life tests are essential for aircraft components in the aerospace industry to ensure safety. Throughout its lifecycle, a product faces various external challenges such as temperature, humidity, wind, and pressure. Alongside these external factors, the product's inherent strength allows it to withstand these stresses. The SSM is now widely used across many fields, especially for tackling reliability assessment challenges. An insightful monograph discussing various SSMs was authored by Kotz et al. [1].

The lifespan of a component with strength W under two stresses U and V is dictated by the SSM. A component can survive if its strength W is greater than stress U and smaller than stress V. The quantity $\Upsilon = P(U < W < V)$ quantifies the reliability of the device within the framework of the mechanical reliability of this model. Systolic and diastolic blood pressure, for instance, have two limits that a person's blood pressure should fall within. Consequently, the quantity Υ, despite being termed the SSM, possesses applications that extend beyond the assessment of actual SSM. It is applicable in a variety of fields, including mechanical design, information engineering, quality control, reliability analysis, and materials science. Chandra and Owen [2] first introduced the methodology's fundamental concept of Υ. In the literature, various studies regarding the model $\Upsilon = P(U < W < V)$ have been conducted by different authors, employing a range of sampling designs and distributions. Ivshin [3] investigated the maximum likelihood estimator (MLE) and the minimal variance unbiased estimator of Υ when both stress and strength are uniform or exponential random variables with an unknown location parameter. With a Weibull distribution, Hassan et al. [4] concentrated on estimating the parameter Υ in the presence of k outliers. The case of nonparametric inference of Υ was covered by Guangming et al. [5]. The estimate of Υ, assuming that the stresses and strength are independent variables that adhere to the inverse Kumaraswamy distribution, was examined, respectively, by Hameed et al. [6] in a complete sample and Hassen et al. [7] in ranked set sampling. In the work of Abd Elfattah and Taha [8], the reliability estimator of Υ based on the inverse Rayleigh distribution was analyzed, taking into account data outliers. Raheem et al. [9] looked into traditional estimation methods, assuming an inverse Rayleigh distribution for both stress and strength random variables. Attia and Karam[10] explored the Bayesian estimation of Υ in the context of a Dagum distribution. Choudhary et al. [11] performed a statistical estimation of Υ within a Weibull distribution framework, utilizing progressively censored data. Yousef and Elmetwally [12] investigated the reliability estimator of Υ based on progressive first failure. For recent studies, the reader can refer to Yousef et al. [13], Hassan et al. [14], Alotaibi et al. [15], Moheb et al. [16] and Hassan and Mogran [17].

Cho et al. [18] presented a novel censoring scheme called the generalized progressive hybrid censoring (GPHC). This plan ensures a sufficient number of failures, which can increase the effectiveness of statistical inference, in addition to ending the experiment within the predetermined testing period. The following is a description of the GPHC.

Assume that our research group consists of n independent units with the same lifetime distribution, where $Z_1, Z_2, ..., Z_n$ represent the corresponding lifetime. The integers k and s, $k < s$, as well as $R_1, R_2, ..., R_s$, which can satisfy the equation $\sum_{i=1}^{s} R_i + s = n$ function as preplanned integers and have been under predetermination between zero and n. On the arrival of the first failure Z_1, we withdraw R_1 units. When the second failure, Z_2, occurs, we remove R_2 units at random from the $n-2-R_1$ survivors. With the rest of the survival units removed, the process is repeated and ended at $\tau^* = \max(\min(\tau, Z_s), Z_k)$. It vastly improved prior approaches by allowing us to choose whether or not to continue the experiment if the sample size is insufficient at the predetermined cut-off time τ. Researchers would prefer to obtain s failures under the GPHC scheme, but they can alternatively choose k failures, which are considered the bare minimum. The GPHC scheme is referred to as $R_1, R_2, ..., R_s$. Let D be the observed failure times before arriving at the predefined time τ.

The GPHC scheme can be classified into the following categories:

$$\begin{array}{l} \text{Case 1: } Z_1, ..., Z_d, ..., Z_k \text{ for } \tau < Z_k < Z_s, \\ \text{Case 2: } Z_1, ..., Z_k, ..., Z_d \text{ for } Z_k < \tau < Z_s, \\ \text{Case 3: } Z_1, ..., Z_k, ..., Z_s \text{ for } Z_k < Z_s < \tau. \end{array}$$

The likelihood function of a random sample with cumulative distribution function (CDF) $F(z)$ under GPHC is as follows.

$$l(\gamma; Z) = \begin{cases} A_1 \prod_{j=1}^{k-1} f(z_{j:s:n})(1 - F(z_{j:s:n}))^{R_j} f(z_{k:s:n})(1 - F(z_{k:s:n}))^{R_k^*} & \text{Case 1}, \\ A_2 \prod_{j=1}^{d} f(z_{j:s:n})(1 - F(z_{j:s:n}))^{R_j} (1 - F(\tau))^{R_{d+1}^*} & \text{Case 2}, \\ A_3 \prod_{j=1}^{s} f(z_{j:s:n})(1 - F(z_{j:s:n}))^{R_j} & \text{Case 3}, \end{cases} \tag{1}$$

where γ is the vector of parameters for the lifetime distribution, $A_1 = \prod_{j=1}^{k} \sum_{k=j}^{s}(R_k + 1), A_2 = \prod_{j=1}^{d} \sum_{k=j}^{s}(R_k + 1), A_3 = \prod_{j=1}^{s} \sum_{k=j}^{s}(R_k + 1), R_k^* = n - k - \sum_{i=1}^{k-1} R_i$ and $R_{d+1}^* = n - d - \sum_{i=1}^{d} R_i$.

Tu and Gui [19] considered the estimation of unknown parameters featured by the Kumaraswamy distribution based on GPHC. Alotaibi et al. [20] discussed reliability analysis of Kavya Manoharan Kumaraswamy distribution under GPHC. Nagy et al. [21] recently employed a GPHC sample from the Burr XII distribution to estimate the unknown parameters, reliability, and hazard functions. The Runge-Kutta technique was employed by Maswadah [22] to enhance the maximum likelihood estimation method. Based on GPHC, Liu and Gui. [23] derived the point and interval estimators for the unknown parameters, reliability, and hazard rate functions of the bathtub model. Abdelwahab et al. [24]

discussed classical and Bayesian inference for the Kavya–Manoharan generalized exponential distribution based on GPHC. Wang et al. [25] discussed a competing risk model for bivariate Kumaraswamy distributed based on GPHC. Hassan et al. [26], [27], and [28] implemented a GPHC for an inverted Topp-Leone distribution, a generalized inverse exponential distribution, and a generalized Lomax distribution, respectively. For more see [29], [30] and [31].

To the best of our knowledge, no prior work has attempted to estimate Υ using an exponentiated Weibull distribution (EWD) based on the GPHC scheme, which is the motivation behind this paper. The importance of the EWD and its extensive use in numerous fields motivate us to address this issue. Furthermore, this study is thought to be a generalization of the Yousef et al. [13] research. Thus, the main driving force behind this can be summed up as follows:

- Establish reliability inferences for $\Upsilon = P(U < W < V)$, assuming that strength (W) and two stresses (U and V) have independent EWD with the same scale parameter.
- Derive the MLE as well as the Bayesian estimator of Υ under symmetric and asymmetric loss functions.
- Construct asymptotic confidence intervals with the help of the delta method and provide Bayesian credible intervals.
- Create percentile bootstrap (Boot-P) and bootstrap-t (Boot-T) intervals.
- Markov chain Monte Carlo (MCMC) methods are used to tackle the intricate integrals found in the Bayesian analysis of the posterior distribution.
- Analyze the data and conduct a simulation study to see how various estimates behave.

We proceed with the reliability formulation calculation in Section 2 of this paper. Section 3 includes the derivation of the MLEs, approximate confidence intervals, Boot-P, and Boot-T intervals of Υ. The Bayesian estimators with varying loss functions are computed in Section 4. In addition, the MCMC algorithm is employed to derive Bayesian estimators and establish the highest posterior density (HPD) intervals in Section 5. Then, in Section 6, real data applications are used to demonstrate data analysis. Conclusions are organized in Section 7.

2. Reliability Formulation

A very adaptable class of probability distribution functions is the three-parameter EWD, which Mudholkar and Srivastava [32] introduced as an extension of the Weibull family. The significance of this distribution arises from the fact that the function of the survival rate takes various forms, making it suitable for studying and covering many problems of time reconciliation and validity. Many probability distributions are included as special cases. Mudholkar et al. [33] provided examples of how the EWD is used in reliability and survival studies. The probability density function (PDF) of the EWD is defined as

$$f(x;\eta,\lambda,\theta)=\eta\lambda\theta x^{\theta-1}e^{-\eta x^{\theta}}(1-e^{-\eta x^{\theta}})^{\lambda-1};\qquad x>0,\quad \eta,\lambda,\theta>0, \tag{2}$$

and the corresponding CDF is as follows:

$$F(x;\eta,\lambda,\theta)=(1-e^{-\eta x^{\theta}})^{\lambda}, \tag{3}$$

where $\lambda>0$ and $\theta>0$ are the shape parameters and $\eta>0$ is the scale parameter. Assuming that η is known parameter ($\eta=1$), then we write $X\sim \text{EWD}(\lambda,\theta)$.

The hazard function has various shapes depending on parameter values, such as monotone increasing ($\theta\geq 1,\theta\lambda\geq 1$), monotone decreasing ($\theta\leq 1,\theta\lambda\leq 1$), or unimodal ($\theta<1,\theta\lambda>1$), see Figure 1 for illustration. These different shapes gave the distribution more flexibility in fitting different real data. When $\lambda=1$, the distribution is referred to as a Weibull distribution, and when $\theta=1$, it reduces to the exponentiated exponential distribution (EED). If $\theta=\lambda=1$, the distribution has an exponential constant hazard function. For additional results and applications, one can see Almalki and Nadarajah [34], Wu and Lee [35], Ahmad et al. [36], Cheema et al. [37], Rahman et al. [38], Xie [39] and Ishag et al. [40].

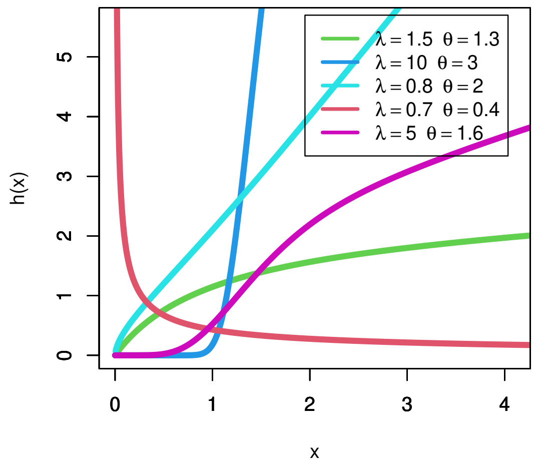

Figure 1: hazard function with various parameters values

Let the random variables $U\sim \text{EWD}(\lambda_1,\theta)$, $W\sim \text{EWD}(\lambda_2,\theta,)$, $V\sim \text{EWD}(\lambda_3,\theta)$ be independent. The reliability formula of the SSM that the probability of a component strength falling in between two stresses is given by:

$$\Upsilon=\int_0^{\infty}\int_u^{\infty}\int_u^{v} f(u;\lambda_1,\theta)f(w;\lambda_2,\theta)f(v;\lambda_3,\theta)dwdvdu,$$

which gets the following formula:

$$\Upsilon = \frac{\lambda_2 \lambda_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_3)}. \tag{4}$$

The reliability Υ of the SSM is shown in Figure 2, and it is clear that the reliability value increases as the value of the parameter λ_3 increases. Also, it is observed that as the parameter values change, the reliability rating does so as well. In most cases, this rating is high and covers the majority of the values.

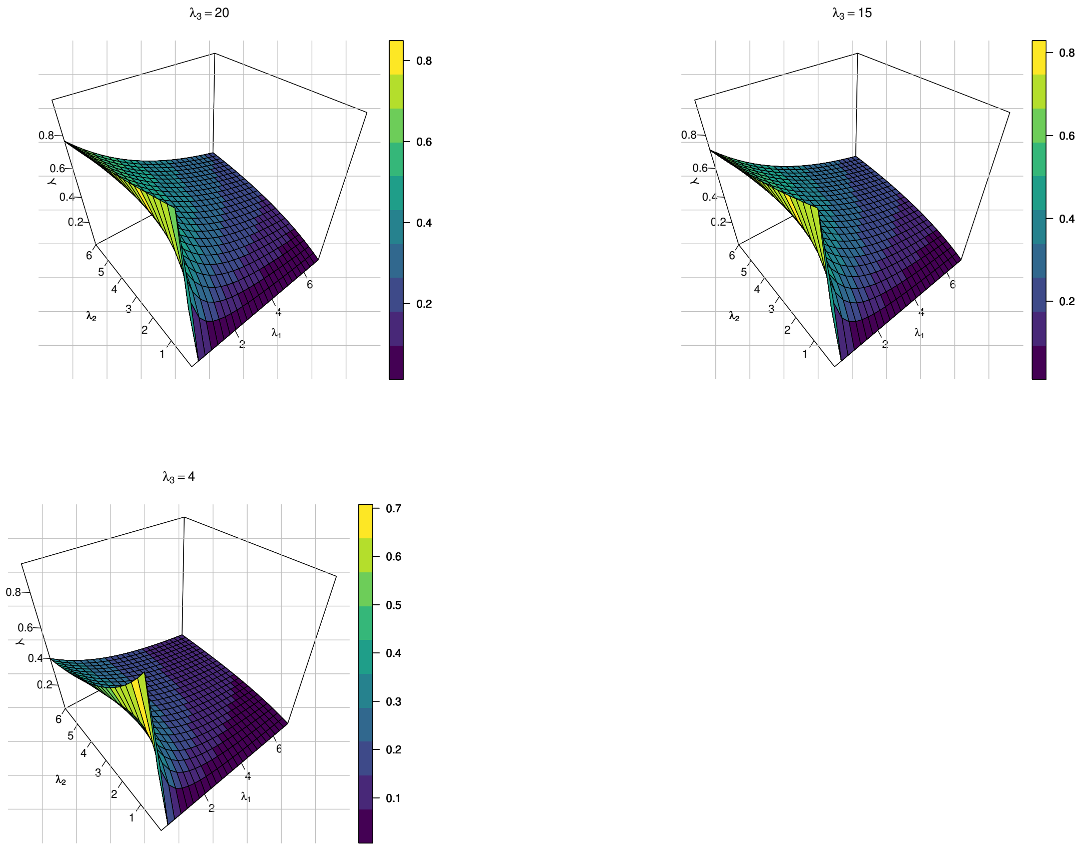

Figure 2: 3D plot of Υ reliability in SSM

3. Maximum Likelihood Estimation

The maximum likelihood (ML) procedure is a popular and effective strategy used by statisticians when dealing with reliability issues and survival analysis. The unknown parameters $\lambda_1, \lambda_2, \lambda_3$, and θ will are estimated using this method to obtain Υ. In order to start we have one of the following types of observations for each identical EWD component stress $U = (u_{1:s_1:n_1}, u_{2:s_1:n_1}, ..., u_{D_1:s_1:n_1}), V = (v_{1:s_3:n_3}, v_{2:s_3:n_3}, ..., v_{D_3 2:s_3:n_3})$ and strength $W = (w_{1:s_2:n_2}, w_{2:s_2:n_2}, ..., w_{D_2 1:s_2:n_2})$ sample under GPHC scheme. Throughout the paper, denote $U = (u_1, ..., u_{D_1})$, where $u_j = u_{j:s_1:n_1}, j = 1, ..., D_1$, similarly for v_j, and w_j.

In our situation, the likelihood function of the observed stress sample U can be obtained by replacing its CDF and PDF in (1) as shown below.

$$l(\lambda_1, \theta; U) = A_1^* \lambda_1^{D_1} \theta^{D_1} \prod_{j=1}^{D_1} u_j^{\theta-1} e^{-u_j^\theta} (1 - e^{-u_j^\theta})^{\lambda_1 - 1} (1 - (1 - e^{-u_j^\theta})^{\lambda_1})^{R_{j1}} (1 - (1 - e^{-\tau_1^\theta})^{\lambda_1})^{R^*_{d1+1}},$$

where

$$D_1 = \begin{cases} k_1 - 1 & \text{Case 1,} \\ d_1 & \text{Case 2,} \\ s_1 & \text{Case 3.} \end{cases}$$

Let $Q = (U, V, W)$ represents the stress and strength samples, and let $\gamma = (\lambda_1, \lambda_2, \lambda_3, \theta)$ be the vector of parameters. The likelihood function of the observed data can be expressed as follows based on the observations of the given data under GPHC scheme.

$$\begin{aligned} l(\gamma; Q) &= A_1^* A_2^* A_3^* \lambda_1^{D_1} \lambda_2^{D_2} \lambda_3^{D_3} \theta^{D_1 + D_2 + D_3} \\ &\times \prod_{j=1}^{D_1} u_j^{\theta-1} e^{-u_j^\theta} \psi_1(u_j, \theta)^{\lambda_1 - 1} (1 - \psi_1(u_j, \theta)^{\lambda_1})^{R_{j1}} (1 - \psi_1(\tau_1, \theta)^{\lambda_1})^{R^*_{d1+1}} \\ &\times \prod_{j=1}^{D_2} w_j^{\theta-1} e^{-w_j^\theta} \psi_2(w_j, \theta)^{\lambda_2 - 1} (1 - \psi_2(w_j, \theta)^{\lambda_2})^{R_{j2}} (1 - \psi_2(\tau_2, \theta)^{\lambda_2})^{R^*_{d2+1}} \\ &\times \prod_{j=1}^{D_3} v_j^{\theta-1} e^{-v_j^\theta} \psi_3(v_j, \theta)^{\lambda_3 - 1} (1 - \psi_3(v_j, \theta)^{\lambda_3})^{R_{j3}} (1 - \psi_3(\tau_3, \theta)^{\lambda_3})^{R^*_{d3+1}}, \end{aligned} \tag{5}$$

where

$$\psi_1(u_j, \theta) = 1 - e^{-u_j^\theta}, \psi_2(w_j, \theta) = 1 - e^{-w_j^\theta}, \psi_3(v_j, \theta) = 1 - e^{-v_j^\theta}, \tag{6}$$

and $\psi_1(\tau_1, \theta)$ is as given by Equation (6) with $u_j = \tau_1$. Similarly for $\psi_2(\tau_2, \theta)$ and $\psi_3(\tau_3, \theta)$. By taking the logarithm of (5), say L, we can get the log-likelihood function as:

$$\begin{aligned} L &= \ln A_1^* A_2^* A_3^* + D_1 \ln \lambda_1 + D_2 \ln \lambda_2 + D_3 \ln \lambda_3 + (D_1 + D_2 + D_3) \ln \theta \\ &- (\theta - 1)[\sum_{j=1}^{D_1} \ln u_j + \sum_{j=1}^{D_2} \ln w_j + \sum_{j=1}^{D_3} \ln v_j] - [\sum_{j=1}^{D_1} u_j^\theta + \sum_{j=1}^{D_2} w_j^\theta + \sum_{j=1}^{D_3} v_j^\theta] \\ &+ (\lambda_1 - 1) \sum_{j=1}^{D_1} \ln \psi_1(u_j, \theta) + (\lambda_2 - 1) \sum_{j=1}^{D_2} \ln \psi_2(w_j, \theta) + (\lambda_3 - 1) \sum_{j=1}^{D_3} \ln \psi_3(v_j, \theta) \end{aligned}$$

$$+\sum_{j=1}^{D_1} R_{j1}\ln(1-(\psi_1(u_j,\theta))^{\lambda_1})+\sum_{j=1}^{D_2} R_{j2}\ln(1-(\psi_2(w_j,\theta))^{\lambda_2})+\sum_{j=1}^{D_3} R_{j3}\ln(1-(\psi_3(v_j,\theta))^{\lambda_3})$$
$$+R^*_{d1+1}\ln(1-(\psi_1(\tau_1,\theta))^{\lambda_1})+R^*_{d2+1}\ln(1-(\psi_2(\tau_2,\theta))^{\lambda_2})+R^*_{d3+1}\ln(1-(\psi_3(\tau_3,\theta))^{\lambda_3}). \tag{7}$$

We take the partial derivatives of (7) for $\lambda_1, \lambda_2, \lambda_3$, and θ respectively, and get a set of likelihood equations as follows:

$$\left.\begin{aligned}
\frac{\partial L}{\partial \lambda_1} &= \frac{D_1}{\lambda_1}+\sum_{j=1}^{D_1}[\ln\psi_1(u_j,\theta)-R_{j1}\vartheta_1(\psi_1(u_j,\theta),\lambda_1)]-R^*_{d1+1}\vartheta_1(\psi_1(\tau_1,\theta),\lambda_1),\\
\frac{\partial L}{\partial \lambda_2} &= \frac{D_2}{\lambda_2}+\sum_{j=1}^{D_2}[\ln\psi_2(w_j,\theta)-R_{j2}\vartheta_2(\psi_2(w_j,\theta),\lambda_2)]-R^*_{d2+1}\vartheta_2(\psi_2(\tau_2,\theta),\lambda_2),\\
\frac{\partial L}{\partial \lambda_3} &= \frac{D_3}{\lambda_3}+\sum_{j=1}^{D_3}[\ln\psi_3(v_j,\theta)-R_{j3}\vartheta_3(\psi_3(v_j,\theta),\lambda_3)]-R^*_{d3+1}\vartheta_3(\psi_3(\tau_3,\theta),\lambda_3),\\
\frac{\partial L}{\partial \theta} &= \frac{D_1+D_2+D_3}{\theta}-[\sum_{j=1}^{D_1}(u_j^\theta+1)\ln u_j+\sum_{j=1}^{D_2}(w_j^\theta+1)\ln w_j+\sum_{j=1}^{D_3}(v_j^\theta+1)\ln v_j]\\
&\quad+(\lambda_1-1)\sum_{j=1}^{D_1}\varphi_1(\psi_1(u_j,\theta),\theta)+(\lambda_2-1)\sum_{j=1}^{D_2}\varphi_2(\psi_2(w_j,\theta),\theta)\\
&\quad+(\lambda_3-1)\sum_{j=1}^{D_3}\varphi_3(\psi_3(v_j,\theta),\theta)-\sum_{j=1}^{D_1}R_{j1}\varpi_1(\psi_1(u_j,\theta),\theta)-\sum_{j=1}^{D_2}R_{j2}\varpi_2(\psi_2(w_j,\theta),\theta)\\
&\quad-\sum_{j=1}^{D_3}R_{j3}\varpi_3(\psi_3(v_j,\theta),\theta)-R^*_{d1+1}\varpi_1(\psi_1(\tau_1,\theta),\theta)-R^*_{d2+1}\varpi_2(\psi_2(\tau_2,\theta),\theta)\\
&\quad-R^*_{d3+1}\varpi_3(\psi_3(\tau_3,\theta),\theta),
\end{aligned}\right\} \tag{8}$$

where

$$\begin{aligned}
\vartheta_i(\psi_i(q_j,\theta),\lambda_i) &= \frac{(\psi_i(q_j,\theta))^{\lambda_i}}{1-(\psi_i(q_j,\theta))^{\lambda_i}}\ln\psi_i(q_j,\theta),\\
\vartheta_i(\psi_i(\tau_i,\theta),\lambda_i) &= \frac{(\psi_i(\tau_i,\theta))^{\lambda_i}}{1-(\psi_i(\tau_i,\theta))^{\lambda_i}}\ln\psi_i(\tau_i,\theta),\\
\varphi_i(\psi_i(q_j,\theta),\theta) &= \frac{q_j^\theta e^{-q_j^\theta}}{\psi_i(q_j,\theta)}\ln q_j,\\
\varpi_i(\psi_i(q_j,\theta),\theta) &= \frac{\lambda_i(\psi_i(q_j,\theta))^{\lambda_i}}{1-(\psi_i(q_j,\theta))^{\lambda_i}}\varphi_i(\psi_i(q_j,\theta),\theta),
\end{aligned}$$

for $q=u,v,w$, and $i=1,2,3$.

Since the equations in (8) are nonlinear, it is obvious that simplifying and obtaining closed-form solutions is difficult. In this situation, Newton's iteration method can be applied to get the MLEs of $\lambda_1, \lambda_2, \lambda_3$, and θ, and then substitute these estimates into Equation (4) to gain the MLE of Υ as:

$$\hat{\Upsilon} = \frac{\hat{\lambda}_2 \hat{\lambda}_3}{(\hat{\lambda}_1 + \hat{\lambda}_2)(\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3)}. \tag{9}$$

3.1. Asymptotic Confidence Intervals

The asymptotic confidence interval (CI) of Υ is determined in this subsection using the asymptotic distribution of $\hat{\Upsilon}$. A $100(1-\zeta)\%$ asymptotic CI of Υ can be constructed as

$$\left(\hat{\Upsilon} - z_{1-\frac{\zeta}{2}}\sqrt{Var(\hat{\Upsilon})}, \hat{\Upsilon} + z_{1-\frac{\zeta}{2}}\sqrt{Var(\hat{\Upsilon})}\right),$$

where $z_{1-\frac{\zeta}{2}}$ is the standard normal variate's upper pth percentile. $Var(\hat{\Upsilon})$ can be obtained by applying a result of Rao [41] (pp. 387), which gives us

$$Var(\hat{\Upsilon}) = \sum_{i=1}^{4} \left(\frac{\partial L}{\partial \gamma_i}\right)^2_{\hat{\gamma}_i} + 2\sum_{i=1}^{2} \left(\frac{\partial L}{\partial \gamma_i}\right)_{\hat{\gamma}_i} \left(\frac{\partial L}{\partial \gamma_j}\right)_{\hat{\gamma}_j} Cov(\hat{\gamma}_i, \hat{\gamma}_j).$$

The observed Fisher information matrix can be used to determine the variance and covariance of $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$, and $\hat{\theta}$. Therefore, the observed Fisher information matrix is given by:

$$\mathbf{I}(\gamma) = [\mathbf{I}_{ij}] = -E\left[\frac{\partial^2 L}{\partial \gamma_i \partial \gamma_j}\right], i, j = 1, 2, 3, 4,$$

where γ is the parameter vector $(\lambda_1, \lambda_2, \lambda_3, \theta)$ with $\gamma_1 = \lambda_1, \gamma_2 = \lambda_2, \gamma_3 = \lambda_3, \gamma_4 = \theta$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain, so it is obtained by dropping the expectation on an operation (see Cohen [42, 43]). The elements of $\mathbf{I}(\gamma)$ are given as follows

$$I_{11} = -\left[\frac{D_1}{\lambda_1^2} + \sum_{j=1}^{D_1} R_{j1} \frac{\partial \vartheta_1(\psi_1(u_j, \theta), \lambda_1)}{\partial \lambda_1} + R^*_{d1+1} \frac{\partial \vartheta_1(\psi_1(\tau_1, \theta), \lambda_1)}{\partial \lambda_1}\right],$$

$$I_{22} = -\left[\frac{D_2}{\lambda_2^2} + \sum_{j=1}^{D_2} R_{j2} \frac{\partial \vartheta_2(\psi_2(w_j, \theta), \lambda_2)}{\partial \lambda_2} + R^*_{d2+1} \frac{\partial \vartheta_2(\psi_2(\tau_2, \theta), \lambda_2)}{\partial \lambda_2}\right],$$

$$I_{33} = -\left[\frac{D_3}{\lambda_3^2} + \sum_{j=1}^{D_3} R_{j3} \frac{\partial \vartheta_3(\psi_3(v_j, \theta), \lambda_3)}{\partial \lambda_3} + R^*_{d3+1} \frac{\partial \vartheta_3(\psi_3(\tau_3, \theta), \lambda_3)}{\partial \lambda_3}\right],$$

$$I_{14} = \sum_{j=1}^{D_1} \left[\varphi_1(\psi_1(u_j,\theta) - R_{j1} \frac{\partial \vartheta_1(\psi_1(u_j,\theta),\lambda_1)}{\partial \theta} \right] - R^*_{d1+1} \frac{\partial \vartheta_1(\psi_1(\tau_1,\theta),\lambda_1)}{\partial \theta},$$

$$I_{24} = \sum_{j=1}^{D_2} \left[\varphi_2(\psi_2(w_j,\theta) - R_{j2} \frac{\partial \vartheta_2(\psi_2(w_j,\theta),\lambda_2)}{\partial \theta} \right] - R^*_{d2+1} \frac{\partial \vartheta_2(\psi_2(\tau_2,\theta),\lambda_2)}{\partial \theta},$$

$$I_{34} = \sum_{j=1}^{D_3} \left[\varphi_3(\psi_3(v_j,\theta) - R_{j3} \frac{\partial \vartheta_3(\psi_3(v_j,\theta),\lambda_3)}{\partial \theta} \right] - R^*_{d3+1} \frac{\partial \vartheta_3(\psi_3(\tau_3,\theta),\lambda_3)}{\partial \theta},$$

$$\begin{aligned} I_{44} = &-\frac{D_1+D_2+D_3}{\theta^2} - [\sum_{j=1}^{D_1} u_j^\theta + \sum_{j=1}^{D_2} w_j^\theta + \sum_{j=1}^{D_3} v_j^\theta](\ln^2\theta + \frac{1}{\theta}) + (\lambda_1 - 1)\sum_{j=1}^{D_1} \frac{\partial \varphi_1(\psi_1(u_j,\theta),\theta)}{\partial \theta} \\ &+ (\lambda_2 - 1)\sum_{j=1}^{D_2} \frac{\partial \varphi_2(\psi_2(w_j,\theta),\theta)}{\partial \theta} + (\lambda_3 - 1)\sum_{j=1}^{D_3} \frac{\partial \varphi_3(\psi_3(v_j,\theta),\theta)}{\partial \theta} - \sum_{j=1}^{D_1} R_{j1} \frac{\partial \varpi_1(\psi_1(u_j,\theta),\theta)}{\partial \theta} \\ &- \sum_{j=1}^{D_2} R_{j2} \frac{\partial \varpi_2(\psi_2(w_j,\theta),\theta)}{\partial \theta} - \sum_{j=1}^{D_3} R_{j3} \frac{\partial \varpi_3(\psi_3(v_j,\theta),\theta)}{\partial \theta} - R^*_{d1+1} \frac{\partial \varpi_1(\psi_1(\tau_1,\theta),\theta)}{\partial \theta} \\ &- R^*_{d2+1} \frac{\partial \varpi_2(\psi_2(\tau_2,\theta),\theta)}{\partial \theta} - R^*_{d3+1} \frac{\partial \varpi_3(\psi_3(\tau_3,\theta),\theta)}{\partial \theta}, \end{aligned}$$

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{32} = I_{23} = 0,$$

where

$$\frac{\partial \vartheta_i(\psi_i(q_j,\theta),\lambda_i)}{\partial \lambda_i} = \frac{(\vartheta_i(\psi_i(q_j,\theta),\lambda_i))^2}{(\psi_i(q_j,\theta))^{\lambda_i}}, \frac{\partial \vartheta_i(\psi_i(\tau_i,\theta),\lambda_i)}{\partial \lambda_i} = \frac{(\vartheta_i(\psi_i(\tau_i,\theta),\lambda_i))^2}{(\psi_i(\tau_i,\theta))^{\lambda_i}},$$

$$\frac{\partial \vartheta_i(\psi_i(q_j,\theta),\lambda_i)}{\partial \theta} = \frac{2\psi_i(\psi_i(q_j,\theta),\theta)}{1-(\psi_i(q_j,\theta))^{\lambda_i}}, \frac{\partial \varphi_i(\psi_i(q_j,\theta),\theta)}{\partial \theta} = \varphi_i(\psi_i(q_j,\theta),\theta)(\ln q_j - e^{q_j^\theta}),$$

$$\frac{\partial \varpi_i(\psi_i(q_j,\theta),\theta)}{\partial \theta} = \frac{\partial \varpi_i(\psi_i(q_j,\theta),\theta)}{\partial \theta} = \frac{\lambda_i \varphi_i'(\psi_i(q_j,\theta),\theta)}{\left[(\psi_i(q_j,\theta))^{-\lambda_i} - 1\right]} + \frac{\lambda_i^2 \varphi_i(\psi_i(q_j,\theta),\theta)\,(\psi_i(q_j,\theta))^{-\lambda_i-1}}{\left[(\psi_i(q_j,\theta))^{-\lambda_i} - 1\right]^2},$$

for $q = u, v, w, i = 1, 2, 3$ and $\varphi_i'(\psi_i(q_j,\theta),\theta) = \frac{(\ln q_j)^2 q_j^\theta e^{-q_j^\theta}\left[\psi_i(q_j,\theta)(1-q_j^\theta)+q_j^\theta e^{-q_j^\theta}\right]}{(\psi_i(q_j,\theta))^2}$.

3.2. Bootstrap Methods

A resampling technique, the bootstrap method for constructing more widely used CIs is discussed in this subsection. Algorithms 1 and 2 present the algorithms for the Boot-P and Boot-T methods, respectively.

- **Boot-P method:**

Algorithm 1 The algorithm of Boot-P method.

Step 1: Generate three random samples from EWD(λ_1, θ), EWD(λ_2, θ), and EWD(λ_3, θ), and use the GPHC scheme to obtain the observed data.
Step 2: Calculate the ML estimates (for example, $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$).
Step 3: Utilize $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$, to generate three new samples, respectively.
Step 4: Obtain new ML estimates $(\hat{\lambda}_1^{*(k)}, \hat{\lambda}_2^{*(k)}, \hat{\lambda}_3^{*(k)})$.
Step 5: Compute the bootstrap estimate of Υ in Step 4 and indicate it by $\hat{\Upsilon}^*$.
Step 6: Repeat Steps 3 and 5 B times.
Step 7: Obtain the results $((\hat{\Upsilon}_1^*, \hat{\Upsilon}_2^*, ..., \hat{\Upsilon}_B^*))$.
Step 8: Define $\hbar(x) = P(\hat{\Upsilon}^* \leq x)$ be the CDF of $\hat{\Upsilon}^*$. Let $\hat{\Upsilon}_{Boot-P}(x) = \hbar^{-1}(x)$ for a given x.
Step 9: The $100(1-\zeta)\%$ symmetric Boot-P CI for Υ is

$$(\hat{\Upsilon}_{Boot-P}(\zeta/2), \hat{\Upsilon}_{Boot-P}(1-\zeta/2)).$$

- **Boot-T method:**

Algorithm 2 The algorithm of Boot-T method.

Steps 1 to 5 are the same as those in Algorithm 1.
Step 6: Define statistic $T_\Upsilon^* = \frac{(\hat{\Upsilon}^* - \hat{\Upsilon})}{\sigma_\Upsilon^*}$.
Step 7: Repeat Steps 3 and 6 B times, then we have $T_\Upsilon^{*(1)}, T_\Upsilon^{*(2)}, ..., T_\Upsilon^{*(B)}$.
Step 8: Define $\aleph(x) = P(T_\Upsilon^* \leq x)$ be the CDF of T_Υ^*. Let $\hat{\Upsilon}_{Boot-T} = \hat{\Upsilon} + \aleph^{-1}(x)\sigma_\Upsilon$.
Step 9: The $100(1-\zeta)\%$ symmetric Boot-T CI for Υ is

$$(\hat{\Upsilon}_{Boot-T}(\zeta/2), \hat{\Upsilon}_{Boot-T}(1-\zeta/2)).$$

4. Bayesian Estimation

In this section, Bayesian estimation of Υ is obtained when data are observed using GPHC based on a squared error loss function (SELF) and a linear exponential (LINEX) loss function, which are defined respectively by

$$L_1 = (\rho, \check{\rho}) = (\check{\rho} - \rho)^2,$$
$$L_2 = (\rho, \check{\rho}) = e^{c(\check{\rho} - \rho)} - c(\check{\rho} - \rho) - 1,$$

where $\check{\rho}$ is an estimator of ρ. Denote the prior and posterior distributions of ρ by $\pi(\rho)$ and $\pi^*(\rho \mid \underline{q})$, respectively. Under the SELF and LINEX loss functions, the Bayesian estimation of any function $B(\rho)$ of ρ is given by

$$\tilde{v}_{SELF} = E[B(\rho) \mid \underline{q}] = \int_0^\infty B(\rho)\pi^*(\rho \mid \underline{q})d\rho,$$
$$\tilde{v}_{LINEX} = \frac{-1}{c} \ln\left(E(e^{-cB(\rho)})\right) = \frac{-1}{c} \ln\left(\int_0^\infty e^{-cB(\rho)}\pi^*(\rho \mid \underline{q})d\rho\right).$$

The prior distribution is important for the development of Bayes estimators. Under the assumption of gamma prior distributions, we investigate this estimation problem. Therefore, it is assumed here that $\lambda_1, \lambda_2, \lambda_3$, and θ follow independent gamma distributions with $\lambda_1 \sim G(a_1, b_1), \lambda_2 \sim G(a_2, b_2), \lambda_3 \sim G(a_3, b_3)$, and $\theta \sim G(a_4, b_4)$ with probability densities given by, respectively,

$$\pi(\lambda_i) = \frac{\lambda_i^{a_i-1}}{\Gamma(a_i) b_i^{a_i}} e^{-\frac{\lambda_i}{b_i}}, \quad \pi(\theta) = \frac{\theta^{a_4-1}}{\Gamma(a_4) b_4^{a_4}} e^{-\frac{\theta}{b_4}}, \quad \lambda_i > 0, a_i, b_i > 0, i = 1, 2, 3. \tag{10}$$

Using the informative prior (10) and the likelihood function (5), the joint posterior density can be derived as follows:

$$\begin{aligned}
\pi^*(\gamma) = & \prod_{i=1}^{3} A_i^* \frac{\lambda_i^{D_i+a_i-1}}{\Gamma(a_i) b_i^{a_i}} \frac{\theta^{a_4-1+\sum_{j=1}^{3} D_j}}{\Gamma(a_4) b_4^{a_4}} e^{-\frac{\theta}{b_4}} \\
& \times e^{-\sum_{j=1}^{D_1}\left[(1-\theta)\ln u_j + u_j^\theta + \frac{\lambda_1}{b_1} - (\lambda_1-1)\ln \psi_1(u_j,\theta) - R_{j1}\ln(1-(\psi_1(u_j,\theta))^{\lambda_1})\right] + R^*_{d1+1}\ln(1-(\psi_1(\tau_1,\theta))^{\lambda_1})} \\
& \times e^{-\sum_{j=1}^{D_2}\left[(1-\theta)\ln w_j + w_j^\theta + \frac{\lambda_2}{b_2} - (\lambda_2-1)\ln \psi_2(w_j,\theta) - R_{j2}\ln(1-(\psi_2(w_j,\theta))^{\lambda_2})\right] + R^*_{d2+1}\ln(1-(\psi_2(\tau_2,\theta))^{\lambda_2})} \\
& \times e^{-\sum_{j=1}^{D_3}\left[(1-\theta)\ln v_j + v_j^\theta + \frac{\lambda_3}{b_3} - (\lambda_3-1)\ln \psi_3(v_j,\theta) - R_{j3}\ln(1-(\psi_3(v_j,\theta))^{\lambda_3})\right] + R^*_{d3+1}\ln(1-(\psi_3(\tau_3,\theta))^{\lambda_3})}.
\end{aligned}$$

The marginal posterior densities of the parameters $\lambda_1, \lambda_2, \lambda_3$, and θ can be derived as

$$\left.\begin{aligned}
\pi^*(\lambda_1) & \propto \lambda_1^{D_1+a_1-1} e^{-\sum_{j=1}^{D_1}\left[\lambda_1(\frac{1}{b_1} - \ln\psi_1(u_j,\theta)) - R_{j1}\ln(1-(\psi_1(u_j,\theta))^{\lambda_1})\right] + R^*_{d1+1}\ln(1-(\psi_1(\tau_1,\theta))^{\lambda_1})}, \\
\pi^*(\lambda_2) & \propto \lambda_2^{D_2+a_2-1} e^{-\sum_{j=1}^{D_2}\left[\lambda_2(\frac{1}{b_2} - \ln\psi_2(w_j,\theta)) - R_{j2}\ln(1-(\psi_2(w_j,\theta))^{\lambda_2})\right] + R^*_{d2+1}\ln(1-(\psi_2(\tau_2,\theta))^{\lambda_2})}, \\
\pi^*(\lambda_3) & \propto \lambda_3^{D_3+a_3-1} e^{-\sum_{j=1}^{D_3}\left[\lambda_3(\frac{1}{b_3} - \ln\psi_3(v_j,\theta)) - R_{j3}\ln(1-(\psi_3(v_j,\theta))^{\lambda_3})\right] + R^*_{d3+1}\ln(1-(\psi_3(\tau_3,\theta))^{\lambda_3})}, \\
\pi^*(\theta) & \propto \theta^{a_4-1+\sum_{i=1}^{3} D_i} e^{-\frac{\theta}{b_4}} \\
& \times e^{-\sum_{j=1}^{D_1}\left[u_j^\theta - \theta\ln u_j - (\lambda_1-1)\ln\psi_1(u_j,\theta) - R_{j1}\ln(1-(\psi_1(u_j,\theta))^{\lambda_1})\right] + R^*_{d1+1}\ln(1-(\psi_1(\tau_1,\theta))^{\lambda_1})} \\
& \times e^{-\sum_{j=1}^{D_2}\left[w_j^\theta - \theta\ln w_j - (\lambda_2-1)\ln\psi_2(w_j,\theta) - R_{j2}\ln(1-(\psi_2(w_j,\theta))^{\lambda_2})\right] + R^*_{d2+1}\ln(1-(\psi_2(\tau_2,\theta))^{\lambda_2})} \\
& \times e^{-\sum_{j=1}^{D_3}\left[v_j^\theta - \theta\ln v_j - (\lambda_3-1)\ln\psi_3(v_j,\theta) - R_{j3}\ln(1-(\psi_3(v_j,\theta))^{\lambda_3})\right] + R^*_{d3+1}\ln(1-(\psi_3(\tau_3,\theta))^{\lambda_3})}.
\end{aligned}\right\} \tag{11}$$

The marginal posterior densities in (11) are not well-known distributions, so we will use the Metropolis-Hastings (MH) sampler to generate the values of $\lambda_1, \lambda_2, \lambda_3$, and θ with a normal proposal distribution to generate samples from it in (11).

Furthermore, the approach of Chen and Shao [44] is extensively used to construct HPD intervals with unknown benefit distribution parameters for Bayesian estimates. For example, a 95% HPD interval can be created using two endpoints from the MCMC sample outputs: 2.5% and 97.5% percentiles, respectively. The Θ parameters' Bayes, trustworthy intervals are calculated as follows:

(i) Sorted parameters as $\tilde{\lambda}_l^{[1]} < \tilde{\lambda}_l^{[2]} < ... < \tilde{\lambda}_l^{[N]};\ l = 1, 2, 3,\ \tilde{\theta}^{[1]} < \tilde{\theta}^{[2]} < ... < \tilde{\theta}^{[N]}$, and $\Upsilon^{[1]} < \Upsilon^{[2]} < ... < \Upsilon^{[N]}$, and N is the length of MCMC generated.

(ii) The 95% symmetric credible intervals of $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\theta}$ and $\tilde{\Upsilon}$ become $\left(\tilde{\lambda}_l^{L\frac{25}{1000}}, \tilde{\lambda}_l^{L\frac{975}{1000}}\right)$, $\left(\tilde{\theta}^{L\frac{25}{1000}}, \tilde{\theta}^{L\frac{975}{1000}}\right)$ and $\left(\tilde{\Upsilon}^{L\frac{25}{1000}}, \tilde{\Upsilon}^{L\frac{975}{1000}}\right)$.

5. Simulation Study

This section presents some simulation findings that demonstrate how the various tactics discussed in this study perform in real-world situations.

- Various circumstances were used as:
  Case I: $\lambda_1 = 0.5, \lambda_2 = 4, \lambda_3 = 20, \theta = 0.5$ and $\tau_1 = 14, \tau_2 = 6, \tau_3 = 18$.
  Case II:$\lambda_1 = 0.5, \lambda_2 = 1.2, \lambda_3 = 10, \theta = 3$ and $\tau_1 = 1, \tau_2 = 1.2, \tau_3 = 1.6$.
  Case III:$\lambda_1 = 0.3, \lambda_2 = 0.6, \lambda_3 = 5, \theta = 0.8$ and $\tau_1 = 2, \tau_2 = 1.8, \tau_3 = 3$.
  Case IV: $\lambda_1 = 1.3, \lambda_2 = 5, \lambda_3 = 15, \theta = 1.5$ and $\tau_1 = 2, \tau_2 = 1.8, \tau_3 = 3$ and $\tau_1 = 2.5, \tau_2 = 2.5, \tau_3 = 4$.

- A variety of sample sizes were selected, $n_1 = 20, n_2 = 25, n_3 = 15$ with different effective sample sizes for each sample as $s_1 = 15, s_2 = 18, s_3 = 11$, and different k values for each sample as $k_1 = 12, k_2 = 16, k_3 = 10$.

- Big sample sizes were selected as $n_1 = 30, n_2 = 40, n_3 = 30$ with different effective sample sizes for each sample as $s_1 = 17, s_2 = 22, s_3 = 13$, and different k values for each sample as $k_1 = 15, k_2 = 20, k_3 = 11$.

- Two different progressive censoring schemes, namely, Scheme-I $R_1 = (n_1 - s_1, rep(0, s_1 - 1))$, $R_2 = (n_2 - s_2, rep(0, s_2 - 1))$, $R_3 = (n_3 - s_3, rep(0, s_3 - 1))$ and Scheme-II $R_1 = (rep(0, s_1 - 1), n_1 - s_1)$, $R_2 = (rep(0, s_2 - 1), n_1 - s_1)$, $R_3 = (rep(0, s_3 - 1), n_3 - s_3)$.

- Calculate the point estimates based on both estimation methods. Also, calculate the credible and approximate 95% CIs for both loss functions in each instance.

- The procedures are carried out 5000 times, and the results are presented for the bias and mean squared errors (MSE) for all estimates.

- The average lengths of CI (LCI) with related coverage percentages (CP), average lengths of Boot-P (LCIBP), average lengths of Boot-T (LCIBT), and average lengths of HPD credible CI (LCCI) where LCCI1 for SELF, LCCI2 for LINEX when c= 0.5 and LCCI3 for LINEX when c=1.5.

Elective hyperparameters based on the mean and variance of the gamma prior distribution are used to choose the hyper-parameters for prior distribution. Using the likelihood method's estimate and variance-covariance matrix, we may learn how to elicit hyperparameters of the independent joint prior. The resulting hyperparameters can be represented as the mean and variance of gamma priors.

$$a_j = \frac{\left[\frac{1}{N}\sum_{i=1}^{N}\hat{\rho_j^i}\right]^2}{\frac{1}{N-1}\sum_{i=1}^{N}\left[\hat{\rho_j^i} - \frac{1}{N}\sum_{i=1}^{N}\hat{\rho_j^i}\right]^2};\ j = 1, 2, ..., p-1,$$

$$b_j = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{\rho_j^i}}{\frac{1}{N-1}\sum_{i=1}^{N}\left[\hat{\rho_j^i} - \frac{1}{N}\sum_{i=1}^{N}\hat{\rho_j^i}\right]^2};\ j = 1, 2, ..., p-1,$$

where N is the number of iterations (for more details see Dey et al. [45]). We replicate the MH algorithm process 10,000 times for MCMC approaches. The following conclusions can be drawn from the results shown in Tables”1, 2, 3, and 4.

- Based on the average (AvE), where they approach to the actual values, the estimates of population parameters for ML and Bayesian methods are pretty good (see Tables 1, 2, 3, and 4).
- The MSE reduces with sample size, as is expected for the ML and Bayesian estimation techniques as seen in Tables 1, 2, 3, and 4.
- As s increases for a given sample size, the MSE likewise gets worse.
- The MSEs fall as the tolerable absolute minimum of failures, k, rises, if n and s are held constant as seen in Table 1 to Table 4.
- Based on the evidence presented in Tables 1- 4, Scheme II demonstrates clear superiority over Scheme I in terms of bias, MSE, length of CI.
- The Bayesian estimates perform better than ML estimates in terms of bias, MSE, and length of CI because they take into account prior information based on a gamma informative prior (see Tables 1-4).
- Bayes estimates derived using asymmetric loss functions are more accurate than those employing symmetric loss functions, as demonstrated in Tables 1-4.
- For Bayesian estimation, the average length of the HPD credible CIs is preferable to the average length of an asymptotic CI. The shortest CI is the average bootstrapping time (see Tables 1-4).

6. Data Analysis

In order to illustrate the approaches suggested in this study, this section looks at a real data set. The EWD based on the GPHC technique is also displayed using this dataset. Chapter Three of Nelson’s book [46] contains the results of a stress-strength life test of transformer insulation. The test included three levels of voltage, which are 35:4kv, 42:4kv, and 46:7kv, respectively, with a normal voltage of 14:4kv.

Table 1: AvE and MSE for Estimation methods: Case I

$\lambda_1 = 0.5, \lambda_2 = 4, \lambda_3 = 20, \theta = 0.5$																			
$\tau_1 = 14, \tau_2 = 6, \tau_3 = 18$					ML		SELF		LINEX c=0.5		LINEX c=1.5								
n_1, n_2, n_3	Scheme	s_1, s_2, s_3	k_1, k_2, k_3		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LACI	CP	LCIBP	LCIBT	LCCI1	LCCI2	LCCI3
20,15,15	1	15,18,11	12,16,10	λ_1	0.0288	0.0173	0.0085	0.0087	0.0067	0.0086	0.0032	0.0085	0.5032	96%	0.0225	0.0224	0.3396	0.3381	0.3351
				λ_2	0.1576	0.4243	-0.0038	0.0193	-0.0065	0.0193	-0.0119	0.0193	2.4799	94%	0.1146	0.1148	0.5392	0.5405	0.5382
				λ_3	0.0455	0.0314	-0.0114	0.0218	-0.0142	0.0220	-0.0197	0.0223	0.6724	95%	0.0319	0.0319	0.5938	0.5880	0.5829
				θ	0.0048	0.0012	0.0024	0.0011	0.0020	0.0011	0.0011	0.0010	0.1327	95%	0.0055	0.0055	0.1297	0.1295	0.1291
				Υ	-0.0072	0.0006	-0.0014	0.0003	-0.0010	0.0003	-0.0004	0.0003	0.0956	96%	0.0044	0.0045	0.0651	0.0649	0.0648
		17,22,13	15,20,11	λ_1	0.0356	0.0163	0.0025	0.0053	0.0017	0.0053	0.0002	0.0053	0.4812	95%	0.0227	0.0228	0.2787	0.2783	0.2783
				λ_2	0.1404	0.4167	0.0039	0.0081	0.0029	0.0081	0.0010	0.0081	2.4722	94%	0.1128	0.1126	0.3422	0.3417	0.3443
				λ_3	0.0425	0.0301	0.0068	0.0081	0.0058	0.0080	0.0038	0.0079	0.6284	99%	0.0308	0.0306	0.3472	0.3450	0.3413
				θ	0.0064	0.0011	0.0048	0.0010	0.0045	0.0010	0.0038	0.0009	0.1278	95%	0.0055	0.0053	0.1334	0.1329	0.1325
				Υ	-0.0086	0.0006	-0.0003	0.0002	-0.0001	0.0002	0.0002	0.0002	0.0915	95%	0.0041	0.0040	0.0534	0.0532	0.0527
	2	15,18,11	12,16,10	λ_1	0.0551	0.0214	0.0124	0.0095	0.0106	0.0093	0.0071	0.0091	0.5320	96%	0.0228	0.0229	0.3717	0.3692	0.3641
				λ_2	0.2231	0.7048	-0.0037	0.0197	-0.0063	0.0198	-0.0115	0.0199	3.1757	95%	0.1536	0.1519	0.5060	0.5048	0.5059
				λ_3	0.0623	0.5909	0.0051	0.0216	0.0024	0.0215	-0.0031	0.0215	3.0064	95%	0.1394	0.1416	0.5883	0.5866	0.5928
				θ	0.0067	0.0014	0.0027	0.0013	0.0022	0.0013	0.0011	0.0013	0.1440	95%	0.0067	0.0067	0.1298	0.1296	0.1293
				Υ	-0.0134	0.0009	-0.0020	0.0003	-0.0016	0.0003	-0.0009	0.0003	0.1040	96%	0.0049	0.0049	0.0712	0.0708	0.0702
		17,22,13	15,20,11	λ_1	0.0363	0.0182	0.0024	0.0055	0.0016	0.0055	0.0000	0.0054	0.5102	96%	0.0228	0.0219	0.2805	0.2798	0.2789
				λ_2	0.1437	0.3905	0.0022	0.0083	0.0012	0.0083	-0.0008	0.0084	2.3865	95%	0.1113	0.1090	0.3542	0.3541	0.3490
				λ_3	0.0144	0.1303	-0.0016	0.0087	-0.0026	0.0088	-0.0047	0.0088	1.4152	99%	0.0601	0.0637	0.3829	0.3823	0.3800
				θ	0.0045	0.0013	0.0037	0.0012	0.0033	0.0012	0.0026	0.0012	0.1384	96%	0.0059	0.0062	0.1371	0.1373	0.1370
				Υ	-0.0088	0.0007	-0.0003	0.0002	-0.0002	0.0002	0.0002	0.0002	0.0958	96%	0.0044	0.0043	0.0534	0.0532	0.0530
30,40,30	1	25,30,22	20,24,20	λ_1	0.0282	0.0105	0.0120	0.0070	0.0104	0.0069	0.0072	0.0067	0.3872	97%	0.0186	0.0185	0.3222	0.3217	0.3200
				λ_2	0.1331	0.0243	0.0102	0.0190	0.0075	0.0199	0.0023	0.0199	1.8612	94%	0.0786	0.0764	0.5319	0.5279	0.5310
				λ_3	0.0068	0.0011	-0.0106	0.0216	-0.0132	0.0217	-0.0184	0.0218	1.4103	98%	0.0662	0.0806	0.5735	0.5757	0.5758
				θ	0.0009	0.0005	0.0000	0.0005	-0.0003	0.0005	-0.0008	0.0005	0.0905	95%	0.0043	0.0043	0.0897	0.0897	0.0896
				Υ	-0.0066	0.0004	-0.0022	0.0003	-0.0019	0.0002	-0.0013	0.0002	0.0731	96%	0.0034	0.0035	0.0613	0.0612	0.0608
		27,34,26	23,30,23	λ_1	0.0234	0.0091	0.0060	0.0048	0.0052	0.0048	0.0037	0.0048	0.4103	96%	0.0193	0.0194	0.2629	0.2622	0.2610
				λ_2	0.0326	0.0235	-0.0042	0.0076	-0.0052	0.0076	-0.0071	0.0077	1.8923	95%	0.0851	0.0690	0.3244	0.3225	0.3220
				λ_3	0.0053	0.0010	-0.0002	0.0008	-0.0012	0.0084	-0.0031	0.0084	0.8675	97%	0.0380	0.0381	0.3506	0.3513	0.3511
				θ	0.0024	0.0005	0.0024	0.0005	0.0022	0.0006	0.0017	0.0004	0.0903	95%	0.0039	0.0038	0.0950	0.0950	0.0950
				Υ	-0.0081	0.0004	-0.0009	0.0002	-0.0008	0.0002	-0.0005	0.0002	0.0761	95%	0.0032	0.0032	0.0506	0.0507	0.0507
	2	25,30,22	20,24,20	λ_1	0.0374	0.0117	0.0143	0.0080	0.0128	0.0079	0.0098	0.0077	0.3974	95%	0.0169	0.0170	0.3298	0.3274	0.3228
				λ_2	0.1411	0.3371	0.0129	0.0194	0.0103	0.0193	0.0051	0.0191	2.2098	95%	0.0976	0.0985	0.5626	0.5598	0.5562
				λ_3	0.0702	0.0009	-0.0027	0.0008	-0.0053	0.0007	-0.0106	0.0006	1.0712	97%	0.0523	0.0534	0.5576	0.5555	0.5481
				θ	0.0015	0.0007	0.0009	0.0007	0.0006	0.0007	0.0000	0.0007	0.1032	95%	0.0048	0.0048	0.1056	0.1057	0.1053
				Υ	-0.0089	0.0004	-0.0026	0.0003	-0.0023	0.0003	-0.0018	0.0003	0.0742	95%	0.0033	0.0034	0.0616	0.0614	0.0612
		27,34,26	23,30,23	λ_1	0.0302	0.0113	0.0057	0.0054	0.0050	0.0054	0.0035	0.0053	0.3999	94%	0.0159	0.0169	0.2829	0.2816	0.2793
				λ_2	0.1151	0.1259	-0.0072	0.0076	-0.0082	0.0077	-0.0102	0.0077	1.9472	96%	0.0885	0.0878	0.3480	0.3489	0.3468
				λ_3	0.0118	0.0008	0.0004	0.0007	-0.0007	0.0007	-0.0027	0.0006	0.9266	99%	0.0410	0.0412	0.3375	0.3362	0.3380
				θ	0.0033	0.0007	0.0028	0.0008	0.0025	0.0006	0.0021	0.0005	0.1023	95%	0.0045	0.0045	0.1047	0.1047	0.1049
				Υ	-0.0069	0.0004	-0.0008	0.0002	-0.0007	0.0002	-0.0004	0.0002	0.0715	94%	0.0032	0.0032	0.0530	0.0529	0.0526

At 42:4kv, the dataset is 0.6, 13.4, 15.2, 19.9, 25.0, 30.2, 32.8, 44.4, 56.2.
At 46:7kv, the dataset is 3.1, 8.3, 8.9, 9.0, 13.6, 14.9, 16.1, 16.9, 21.3,48.1.
At 35:4kv, the dataset is 40.1, 59.4, 71.2, 166.5, 204.7, 229.7, 308.3, and 537.9.
We wish to determine the SSM $P(U < W < V)$ dependability. First, Table 5 presents estimated values for several measures as: 'Kolmogorov Smirnov distance (KS) statistic along with its P-value (P-V), Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected AIC (CAIC), and Hannan-Quinn information criterion (HQIC)' for the EWD. The EWD fits each data set according to the KS test. The ML estimates via a complete sample are listed in Table 6 for EWD and EED with stress-strength Υ. According to the value of different measures and P-V of Υ, it is noted that the EWD is better than the EED. Also, the SE for the ML estimate is smaller than that of the other estimates. In light of the GPHC, the Bayesian estimation method represents the most accurate for the EWD. The Bayesian estimating method's reliability is higher than the ML process's, supporting the stated result. Figures 3, 4, and 5, prove the fitting of each data set fitted by EWD, according to empirical CDF, histogram, and probability-probability (P-P) behavior plots.

Table 2: Bias, MSE, and LCI for Estimation methods: Case II

$\lambda_1 = 0.5, \lambda_2 = 1.2, \lambda_3 = 10, \theta = 3$																			
$\tau_1 = 1, \tau_2 = 1.2, \tau_3 = 1.6$					ML		SELF		LINEX c=0.5		LINEX c=1.5		ML				Bayesian		
n_1, n_2, n_3	Scheme	s_1, s_2, s_3	k_1, k_2, k_3		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LACI	CP	LCIBP	LCIBT	LCCI1	LCCI2	LCCI3
20,25,15	1	15,18,11	12,16,10	λ_1	0.0193	0.0155	0.0128	0.0095	0.0111	0.0094	0.0079	0.0092	0.4826	95%	0.0231	0.0230	0.3574	0.3550	0.3515
				λ_2	0.0498	0.0708	-0.0011	0.0188	-0.0036	0.0187	-0.0087	0.0187	1.0260	96%	0.0476	0.0480	0.5286	0.5272	0.5268
				λ_3	1.1425	7.8730	-0.0019	0.0238	-0.0047	0.0238	-0.0105	0.0239	10.0560	96%	0.4620	0.4721	0.5673	0.5671	0.5658
				θ	0.1315	0.1374	0.0015	0.0137	-0.0006	0.0137	-0.0047	0.0138	1.3597	95%	0.0637	0.0630	0.4445	0.4448	0.4449
				Υ	0.0003	0.0029	-0.0049	0.0017	-0.0045	0.0017	-0.0038	0.0017	0.2119	95%	0.0095	0.0097	0.1533	0.1535	0.1541
		17,22,13	15,20,11	λ_1	0.0170	0.0149	0.0059	0.0053	0.0051	0.0052	0.0034	0.0052	0.4509	95%	0.0226	0.0226	0.2665	0.2674	0.2695
				λ_2	0.0430	0.0690	0.0003	0.0073	-0.0007	0.0073	-0.0025	0.0073	1.0168	96%	0.0489	0.0474	0.3184	0.3180	0.3178
				λ_3	1.0793	4.7472	0.0066	0.0080	0.0056	0.0080	0.0037	0.0080	10.0673	96%	0.4718	0.4477	0.3657	0.3658	0.3634
				θ	0.0811	0.1010	-0.0009	0.0067	-0.0019	0.0067	-0.0037	0.0067	1.2054	95%	0.0572	0.0570	0.3174	0.3172	0.3150
				Υ	-0.0003	0.0028	-0.0017	0.0009	-0.0015	0.0009	-0.0010	0.0009	0.2114	96%	0.0099	0.0093	0.1136	0.1141	0.1149
	2	15,18,11	12,16,10	λ_1	0.0299	0.0181	0.0099	0.0098	0.0082	0.0097	0.0047	0.0095	0.5152	97%	0.0247	0.0246	0.3533	0.3525	0.3527
				λ_2	0.0642	0.0853	-0.0010	0.0180	-0.0035	0.0180	-0.0086	0.0182	1.1182	95%	0.0512	0.0504	0.5110	0.5130	0.5080
				λ_3	0.7819	6.4649	0.0077	0.0235	0.0048	0.0233	-0.0011	0.0231	9.4937	96%	0.4457	0.4583	0.6016	0.6044	0.6026
				θ	0.0913	0.1337	0.0034	0.0154	0.0011	0.0154	-0.0036	0.0153	1.3895	94%	0.0621	0.0624	0.4865	0.4826	0.4787
				Υ	-0.0054	0.0033	-0.0034	0.0017	-0.0030	0.0017	-0.0022	0.0017	0.2254	96%	0.0110	0.0108	0.1481	0.1481	0.1474
		17,22,13	15,20,11	λ_1	0.0230	0.0160	-0.0013	0.0052	-0.0021	0.0052	-0.0037	0.0052	0.4888	95%	0.0220	0.0215	0.2630	0.2625	0.2628
				λ_2	0.0750	0.0843	0.0031	0.0081	0.0022	0.0081	0.0002	0.0081	1.1006	94%	0.0489	0.0503	0.3410	0.3388	0.3393
				λ_3	0.9118	3.0464	-0.0007	0.0073	-0.0017	0.0073	-0.0038	0.0073	7.2292	96%	0.4381	0.4355	0.3225	0.3224	0.3244
				θ	0.0914	0.1008	0.0011	0.0075	0.0002	0.0075	-0.0017	0.0075	1.1930	95%	0.0519	0.0530	0.3267	0.3266	0.3260
				Υ	0.0006	0.0028	0.0013	0.0010	0.0015	0.0010	0.0019	0.0010	0.2070	95%	0.0103	0.0101	0.1155	0.1155	0.1147
30,40,30	1	25,30,22	20,24,20	λ_1	0.0131	0.0087	0.0166	0.0081	0.0150	0.0079	0.0116	0.0076	0.3624	95%	0.0157	0.0157	0.3287	0.3232	0.3182
				λ_2	0.0421	0.0437	0.0080	0.0159	0.0056	0.0158	0.0009	0.0156	0.8036	95%	0.0364	0.0361	0.4913	0.4876	0.4838
				λ_3	0.9674	4.0388	-0.0012	0.0228	-0.0041	0.0229	-0.0099	0.0232	6.9122	96%	0.3122	0.3125	0.6000	0.6053	0.6134
				θ	0.0804	0.0559	0.0097	0.0124	0.0077	0.0123	0.0037	0.0123	0.8725	96%	0.0397	0.0396	0.4200	0.4196	0.4182
				Υ	0.0031	0.0016	-0.0056	0.0014	-0.0052	0.0013	-0.0044	0.0013	0.1582	95%	0.0067	0.0067	0.1334	0.1325	0.1313
		27,34,26	23,30,23	λ_1	0.0159	0.0079	0.0069	0.0047	0.0062	0.0047	0.0047	0.0046	0.3442	95%	0.0143	0.0144	0.2593	0.2584	0.2555
				λ_2	0.0450	0.0391	0.0013	0.0066	0.0004	0.0066	-0.0014	0.0066	0.7553	96%	0.0339	0.0341	0.3140	0.3159	0.3185
				λ_3	0.8746	3.3364	0.0064	0.0080	0.0053	0.0080	0.0033	0.0080	4.2210	94%	0.3042	0.3041	0.3275	0.3269	0.3252
				θ	0.0767	0.0519	0.0068	0.0066	0.0059	0.0066	0.0042	0.0066	0.8190	95%	0.0342	0.0314	0.3236	0.3232	0.3205
				Υ	0.0026	0.0016	-0.0021	0.0008	-0.0019	0.0008	-0.0015	0.0008	0.1565	95%	0.0065	0.0064	0.1084	0.1084	0.1084
	2	25,30,22	20,24,20	λ_1	0.0255	0.0116	0.0096	0.0069	0.0079	0.0068	0.0047	0.0066	0.4099	95%	0.0191	0.0191	0.2993	0.2989	0.2972
				λ_2	0.0562	0.0617	0.0117	0.0191	0.0092	0.0190	0.0044	0.0188	0.9491	95%	0.0430	0.0434	0.5359	0.5350	0.5368
				λ_3	0.8660	4.2109	-0.0091	0.0225	-0.0121	0.0226	-0.0180	0.0229	7.3000	95%	0.3315	0.3315	0.6000	0.5992	0.6038
				θ	0.0775	0.0794	0.0026	0.0130	0.0004	0.0129	-0.0038	0.0129	1.0628	94%	0.0490	0.0487	0.4419	0.4416	0.4449
				Υ	-0.0020	0.0022	-0.0030	0.0012	-0.0026	0.0012	-0.0019	0.0012	0.1832	96%	0.0080	0.0078	0.1322	0.1317	0.1309
		27,34,26	23,30,23	λ_1	0.0194	0.0114	0.0042	0.0054	0.0034	0.0054	0.0018	0.0053	0.3541	95%	0.0152	0.0182	0.2849	0.2851	0.2843
				λ_2	0.0584	0.0510	-0.0008	0.0080	-0.0018	0.0080	-0.0038	0.0080	0.8556	97%	0.0384	0.0382	0.3434	0.3462	0.3461
				λ_3	0.6843	1.6573	0.0079	0.0081	0.0068	0.0080	0.0046	0.0080	7.0074	95%	0.2970	0.2985	0.3455	0.3456	0.3460
				θ	0.0612	0.0579	0.0058	0.0070	0.0049	0.0070	0.0031	0.0069	0.9127	95%	0.0389	0.0389	0.3182	0.3170	0.3151
				Υ	0.0001	0.0020	-0.0012	0.0010	-0.0010	0.0010	-0.0006	0.0010	0.1762	95%	0.0078	0.0068	0.1205	0.1207	0.1212

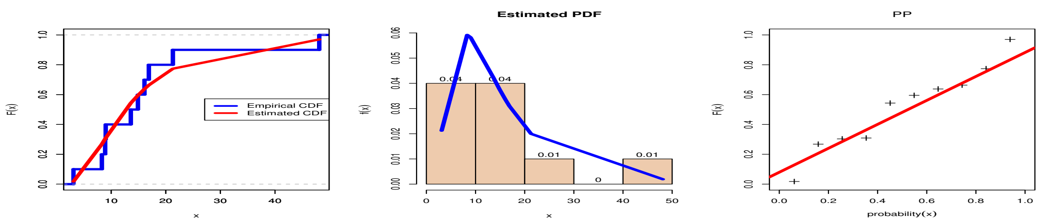

Figure 3: Empirical CDF, histogram, and P-P plots for the EWD for data set 1

Figure 6 discussed the profile likelihood of the EWD parameters, which index the estimators' maximum log-likelihood values. Figure 7 discussed count-our plots of log-likelihood values with EWD parameters which are indicated by the estimators' uniqueness.

The posterior distribution for the MCMC estimate of the SSM for the EWD based on the GPHC is shown in Figure 8 along with its trace and normal curve. Figure 10 presents the MCMC samples as a pairs plot, which shows the pairwise correlation between parameters in the top plot, correlation coefficients in the bottom plot, and marginal frequency distribution for each parameter on the diagonal. Additionally, as seen in Figure 9,

Table 3: Bias, MSE, and LCI for Estimation methods: Case III

$\lambda_1 = 0.3, \lambda_2 = 0.6, \lambda_3 = 5, \theta = 0.8$																			
$\tau_1 = 2, \tau_2 = 1.8, \tau_3 = 3$					ML		SELF		LINEX c=0.5		LINEX c=1.5		ML				Bayesian		
n_1, n_2, n_3	Scheme	s_1, s_2, s_3	k_1, k_2, k_3		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LACI	CP	LCIBP	LCIBT	LCCI1	LCCI2	LCCI3
20,25,15	1	15,18,11	12,16,10	λ_1	0.0082	0.0078	0.0196	0.0062	0.0183	0.0060	0.0156	0.0058	0.3461	95%	0.0148	0.0145	0.2770	0.2757	0.2731
				λ_2	0.0091	0.0254	0.0076	0.0120	0.0055	0.0119	0.0015	0.0117	0.6240	95%	0.0268	0.0269	0.4186	0.4188	0.4189
				λ_3	0.3314	1.8832	-0.0060	0.0209	-0.0086	0.0211	-0.0138	0.0213	5.2253	96%	0.2362	0.2366	0.5804	0.5804	0.5847
				θ	0.0484	0.0256	0.0063	0.0081	0.0047	0.0081	0.0014	0.0080	0.5990	94%	0.0264	0.0263	0.3331	0.3323	0.3296
				Υ	-0.0048	0.0040	-0.0126	0.0033	-0.0123	0.0033	-0.0115	0.0032	0.2477	94%	0.0110	0.0111	0.2177	0.2171	0.2177
		17,22,13	15,20,11	λ_1	0.0112	0.0068	-0.0002	0.0038	-0.0009	0.0038	-0.0022	0.0038	0.3202	95%	0.0144	0.0142	0.2323	0.2315	0.2283
				λ_2	0.0142	0.0248	-0.0034	0.0058	-0.0043	0.0058	-0.0060	0.0058	0.6148	96%	0.0285	0.0283	0.2935	0.2928	0.2915
				λ_3	0.2902	1.6877	0.0059	0.0076	0.0049	0.0076	0.0030	0.0076	4.9689	97%	0.2380	0.2384	0.3390	0.3395	0.3378
				θ	0.0359	0.0206	0.0048	0.0048	0.0040	0.0047	0.0025	0.0047	0.5454	94%	0.0255	0.0257	0.2535	0.2537	0.2544
				Υ	-0.0067	0.0038	0.0002	0.0023	0.0005	0.0023	0.0010	0.0023	0.2417	95%	0.0112	0.0113	0.1803	0.1801	0.1801
	2	15,18,11	12,16,10	λ_1	0.0081	0.0084	0.0134	0.0065	0.0121	0.0063	0.0093	0.0060	0.3578	95%	0.0153	0.0154	0.2818	0.2793	0.2762
				λ_2	0.0085	0.0278	-0.0062	0.0139	-0.0084	0.0138	-0.0126	0.0136	0.6536	95%	0.0285	0.0282	0.4266	0.4253	0.4255
				λ_3	0.3314	1.9484	-0.0040	0.0230	-0.0068	0.0231	-0.0124	0.0233	5.3205	96%	0.2465	0.2366	0.5550	0.5529	0.5541
				θ	0.0454	0.0211	0.0122	0.0097	0.0104	0.0096	0.0067	0.0094	0.5413	96%	0.0236	0.0229	0.3715	0.3714	0.3683
				Υ	-0.0059	0.0043	-0.0113	0.0034	-0.0110	0.0034	-0.0102	0.0033	0.2573	96%	0.0111	0.0111	0.2102	0.2107	0.2077
		17,22,13	15,20,11	λ_1	0.0181	0.0082	0.0041	0.0041	0.0035	0.0041	0.0021	0.0040	0.3483	95%	0.0147	0.0148	0.2394	0.2391	0.2378
				λ_2	0.0287	0.0265	0.0004	0.0069	-0.0004	0.0069	-0.0022	0.0068	0.6289	97%	0.0272	0.0272	0.3203	0.3194	0.3150
				λ_3	0.3534	1.7370	-0.0012	0.0081	-0.0022	0.0081	-0.0043	0.0081	4.9821	96%	0.2196	0.2212	0.3555	0.3566	0.3577
				θ	0.0253	0.0184	0.0047	0.0056	0.0040	0.0056	0.0024	0.0056	0.5223	94%	0.0224	0.0214	0.2910	0.2913	0.2905
				Υ	-0.0052	0.0042	-0.0023	0.0025	-0.0021	0.0025	-0.0015	0.0025	0.2468	95%	0.0110	0.0103	0.1845	0.1851	0.1866
30,40,30	1	28,30,28	18,22,22	λ_1	0.0081	0.0042	0.0128	0.0037	0.0118	0.0037	0.0098	0.0035	0.2508	95%	0.0113	0.0112	0.2233	0.2218	0.2178
				λ_2	0.0179	0.0138	0.0095	0.0092	0.0078	0.0090	0.0044	0.0089	0.4561	95%	0.0190	0.0190	0.3643	0.3640	0.3636
				λ_3	0.3321	0.6837	0.0020	0.0211	-0.0008	0.0211	-0.0063	0.0211	2.9713	95%	0.1332	0.1349	0.5546	0.5509	0.5515
				θ	0.0278	0.0085	0.0081	0.0052	0.0069	0.0052	0.0044	0.0051	0.3439	96%	0.0153	0.0153	0.2760	0.2757	0.2782
				Υ	0.0006	0.0022	-0.0077	0.0021	-0.0074	0.0021	-0.0068	0.0021	0.1834	97%	0.0079	0.0079	0.1712	0.1709	0.1709
		25,35,35	22,30,30	λ_1	0.0057	0.0039	0.0083	0.0032	0.0077	0.0031	0.0064	0.0030	0.2440	95%	0.0110	0.0106	0.2069	0.2060	0.2022
				λ_2	0.0135	0.0115	0.0072	0.0050	0.0065	0.0050	0.0049	0.0050	0.4173	96%	0.0187	0.0191	0.2734	0.2718	0.2716
				λ_3	0.2902	0.6492	0.0060	0.0079	0.0050	0.0079	0.0030	0.0078	3.0593	96%	0.1241	0.1421	0.3382	0.3369	0.3343
				θ	0.0306	0.0077	0.0049	0.0036	0.0043	0.0036	0.0030	0.0036	0.3221	96%	0.0150	0.0156	0.2283	0.2288	0.2295
				Υ	0.0016	0.0021	-0.0038	0.0017	-0.0035	0.0017	-0.0030	0.0016	0.1800	95%	0.0077	0.0077	0.1514	0.1512	0.1495
	2	28,30,28	18,22,22	λ_1	0.0161	0.0060	0.0180	0.0052	0.0168	0.0051	0.0144	0.0049	0.2979	96%	0.0139	0.0140	0.2627	0.2620	0.2612
				λ_2	0.0230	0.0150	0.0122	0.0110	0.0104	0.0109	0.0068	0.0106	0.4720	94%	0.0223	0.0221	0.3958	0.3897	0.3834
				λ_3	0.3658	1.0361	-0.0004	0.0222	-0.0034	0.0223	-0.0093	0.0223	3.7274	95%	0.1684	0.1759	0.5571	0.5613	0.5589
				θ	0.0272	0.0098	0.0105	0.0062	0.0090	0.0061	0.0061	0.0060	0.3737	95%	0.0173	0.0170	0.3030	0.3012	0.3014
				Υ	-0.0025	0.0027	-0.0102	0.0026	-0.0099	0.0025	-0.0092	0.0025	0.2030	96%	0.0086	0.0086	0.1905	0.1900	0.1899
		25,35,35	22,30,30	λ_1	0.0095	0.0042	0.0105	0.0031	0.0099	0.0031	0.0086	0.0030	0.2527	95%	0.0109	0.0109	0.2035	0.2019	0.2014
				λ_2	0.0255	0.0152	-0.0007	0.0050	-0.0015	0.0050	-0.0030	0.0049	0.4736	95%	0.0225	0.0225	0.2685	0.2683	0.2683
				λ_3	0.3813	0.9327	-0.0045	0.0075	-0.0055	0.0075	-0.0075	0.0075	3.4817	95%	0.1526	0.1498	0.3194	0.3201	0.3195
				θ	0.0283	0.0081	0.0084	0.0040	0.0077	0.0039	0.0064	0.0039	0.3356	95%	0.0149	0.0147	0.2323	0.2319	0.2320
				Υ	0.0016	0.0022	-0.0072	0.0017	-0.0070	0.0017	-0.0065	0.0017	0.1855	95%	0.0078	0.0079	0.1558	0.1558	0.1551

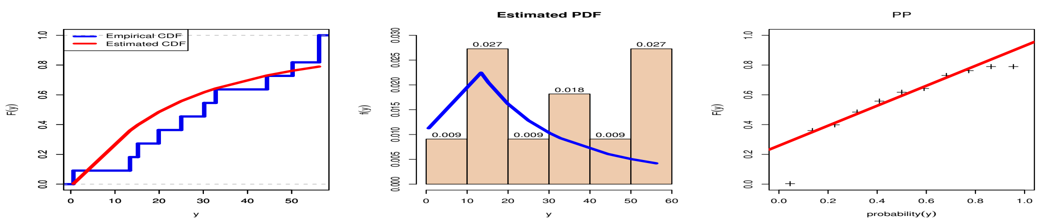

Figure 4: Empirical CDF, histogram, and P-P plots for the EWD for data set 2

convergence starts at 2000 iterations or less for the SSM estimate for EWD based on the entire sample. Figures 13 and 12 show the MCMC samples as a pairs plot that represents the pairwise relationship between parameters as independent with the scatter plot matrix in the top plot, correlation coefficients in the bottom plot, and marginal frequency distribution for each parameter on the diagonal. In this picture, the parameters p3 and p4 are shown to be medially related, where p1 is a λ_1, p2 is a λ_2, p3 is a λ_3, and p4 is a θ.

For each component of this model, we suggested using the following GPHC sample as follows:

Table 4: Bias, MSE, and LCI for Estimation methods: Case IV

$\lambda_1 = 1.3, \lambda_2 = 5, \lambda_3 = 15, \theta = 1.5$																			
$n_1 = 30, n_2 = 40, n_3 = 40$					ML		SELF		LINEX c=0.5		LINEX c=1.5		ML				Bayesian		
τ_1, τ_2, τ_3	Scheme	s_1, s_2, s_3	k_1, k_2, k_3		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LACI	CP	LCIBP	LCIBT	LCCI1	LCCI2	LCCI3
2,1.8,3	1	20,28,28	18,22,22	λ_1	0.0436	0.0673	0.0268	0.0132	0.0169	0.0124	-0.0025	0.0114	1.0034	95%	0.0455	0.0460	0.4201	0.4178	0.4044
				λ_2	0.1569	0.7586	0.0122	0.0787	-0.0201	0.0779	-0.0825	0.0842	3.3618	96%	0.1516	0.1512	1.0791	1.0632	1.0881
				λ_3	0.1916	1.8396	-0.0229	0.5429	-0.0932	0.5571	-0.2204	0.6113	5.2688	94%	0.2449	0.2454	2.8574	2.7490	2.6732
				θ	0.0118	0.0062	0.0025	0.0044	0.0003	0.0044	-0.0040	0.0044	0.3049	95%	0.0141	0.0142	0.2574	0.2570	0.2553
				Υ	-0.0055	0.0012	-0.0037	0.0003	-0.0033	0.0003	-0.0023	0.0002	0.1336	95%	0.0062	0.0062	0.0584	0.0569	0.0560
		25,35,35	22,30,30	λ_1	0.0588	0.0617	0.0288	0.0121	0.0210	0.0120	0.0056	0.0109	0.9804	96%	0.0448	0.0472	0.5470	0.5410	0.5312
				λ_2	0.1264	0.7014	-0.0015	0.0749	-0.0166	0.0729	-0.0462	0.0817	3.2487	95%	0.1430	0.1370	1.1184	1.1026	1.0797
				λ_3	0.1802	1.2887	0.0000	0.1828	-0.0226	0.1841	-0.0664	0.1901	4.8852	94%	0.2367	0.2357	1.6691	1.6675	1.6510
				θ	0.0074	0.0051	0.0015	0.0037	-0.0003	0.0037	-0.0038	0.0037	0.2794	96%	0.0124	0.0124	0.2292	0.2278	0.2265
				Υ	-0.0068	0.0011	-0.0033	0.0003	-0.0027	0.0003	-0.0013	0.0003	0.1372	96%	0.0058	0.0058	0.0625	0.0616	0.0619
	2	20,28,28	18,22,22	λ_1	0.0583	0.0914	0.0257	0.0127	0.0147	0.0119	-0.0067	0.0109	1.1637	96%	0.0501	0.0501	0.4224	0.4134	0.3968
				λ_2	0.1521	0.8848	0.0049	0.0707	-0.0277	0.0703	-0.0903	0.0774	3.6424	97%	0.1535	0.1535	1.0520	1.0518	1.0699
				λ_3	0.2225	2.9436	0.0244	0.5724	-0.0503	0.5641	-0.1839	0.5905	6.6753	95%	0.2892	0.2902	2.9323	2.8524	2.7197
				θ	0.0128	0.0078	0.0046	0.0049	0.0022	0.0049	-0.0026	0.0048	0.3419	95%	0.0151	0.0152	0.2682	0.2675	0.2660
				Υ	-0.0071	0.0014	-0.0030	0.0002	-0.0025	0.0002	-0.0013	0.0002	0.1437	94%	0.0068	0.0070	0.0564	0.0563	0.0559
		25,35,35	22,30,30	λ_1	0.0552	0.0900	0.0261	0.0129	0.0180	0.0109	0.0021	0.0108	1.1571	96%	0.0495	0.0495	0.4945	0.5197	0.5101
				λ_2	0.2528	0.8691	0.0063	0.0681	-0.0087	0.0681	-0.0382	0.0618	3.6131	94%	0.1508	0.1529	1.0891	1.0761	1.0816
				λ_3	0.1178	1.8800	0.0102	0.1480	-0.0109	0.1484	-0.0521	0.1533	5.3603	95%	0.2305	0.2290	1.5367	1.5377	1.4936
				θ	0.0087	0.0053	0.0019	0.0040	0.0001	0.0040	-0.0037	0.0040	0.2840	96%	0.0133	0.0130	0.2419	0.2420	0.2424
				Υ	-0.0076	0.0013	-0.0029	0.0002	-0.0021	0.0003	-0.0007	0.0002	0.1390	96%	0.0055	0.0057	0.0598	0.0598	0.0612
2.5,2.5,4	1	20,28,28	18,22,22	λ_1	0.0416	0.0659	0.0117	0.0132	0.0095	0.0131	0.0050	0.0130	1.0038	95%	0.0421	0.0432	0.4289	0.4290	0.4309
				λ_2	0.1320	0.7271	-0.0161	0.0220	-0.0188	0.0222	-0.0241	0.0227	3.4061	95%	0.1578	0.1571	0.5741	0.5771	0.5809
				λ_3	0.1546	1.6025	-0.0066	0.0230	-0.0095	0.0232	-0.0153	0.0235	5.5506	95%	0.2547	0.2547	0.5874	0.5837	0.5918
				θ	0.0178	0.0061	0.0095	0.0044	0.0084	0.0044	0.0060	0.0043	0.2992	95%	0.0136	0.0133	0.2610	0.2612	0.2603
				Υ	-0.0080	0.0011	-0.0012	0.0002	-0.0010	0.0002	-0.0005	0.0002	0.1261	95%	0.0058	0.0058	0.0498	0.0498	0.0506
		25,35,35	22,30,30	λ_1	0.0409	0.0624	-0.0006	0.0069	-0.0015	0.0069	-0.0033	0.0069	0.9773	95%	0.0420	0.0424	0.3138	0.3137	0.3120
				λ_2	0.1143	0.6581	0.0070	0.0081	0.0060	0.0081	0.0040	0.0080	3.1516	95%	0.1392	0.1445	0.3528	0.3523	0.3511
				λ_3	0.1512	1.1030	0.0018	0.0085	0.0008	0.0084	-0.0011	0.0084	5.5025	95%	0.2440	0.2438	0.3480	0.3471	0.3481
				θ	0.0097	0.0044	0.0022	0.0033	0.0016	0.0033	0.0004	0.0033	0.2589	95%	0.0130	0.0128	0.2213	0.2208	0.2199
				Υ	-0.0040	0.0010	0.0002	0.0001	0.0003	0.0001	0.0005	0.0001	0.1258	95%	0.0056	0.0056	0.0357	0.0357	0.0356
	2	20,28,28	18,22,22	λ_1	0.0315	0.0862	-0.0093	0.0126	-0.0115	0.0157	-0.0159	0.0158	1.1455	95%	0.0523	0.0523	0.4861	0.4858	0.4857
				λ_2	0.1702	0.8386	-0.0208	0.0219	-0.0235	0.0221	-0.0290	0.0226	3.5308	95%	0.1509	0.1506	0.5741	0.5742	0.5848
				λ_3	0.0818	2.0260	-0.0086	0.0253	-0.0117	0.0254	-0.0180	0.0258	5.5759	95%	0.2527	0.2540	0.6038	0.6118	0.6167
				θ	0.0136	0.0061	0.0027	0.0045	0.0014	0.0054	-0.0012	0.0054	0.3021	96%	0.0133	0.0133	0.2822	0.2813	0.2818
				Υ	-0.0049	0.0014	0.0012	0.0002	0.0015	0.0002	0.0019	0.0002	0.1430	95%	0.0063	0.0061	0.0567	0.0560	0.0561
		25,35,35	22,30,30	λ_1	0.0524	0.0754	0.0008	0.0070	-0.0001	0.0070	-0.0020	0.0070	1.0576	97%	0.0473	0.0472	0.3323	0.3330	0.3302
				λ_2	0.1255	0.8078	0.0048	0.0079	0.0038	0.0079	0.0018	0.0079	2.6264	96%	0.1464	0.1473	0.3296	0.3299	0.3308
				λ_3	0.0900	1.9416	0.0088	0.0093	0.0077	0.0092	0.0056	0.0091	4.0894	93%	0.2170	0.2171	0.3673	0.3665	0.3669
				θ	0.0091	0.0049	0.0018	0.0035	0.0012	0.0035	0.0000	0.0035	0.2729	94%	0.0124	0.0124	0.2251	0.2252	0.2247
				Υ	-0.0080	0.0014	0.0001	0.0001	0.0002	0.0001	0.0004	0.0001	0.1408	96%	0.0064	0.0063	0.0389	0.0388	0.0389

Table 5: ML estimates for parameters of EWD with different measures

			Estimates	SE	KS	P-V	AIC	CAIC	BIC	HQIC
u	EWD	λ_1	20.6859	9.2567	0.1680	0.8978	75.6779	77.3921	76.2830	75.0140
		θ	0.4841	0.0557						
w	EWD	λ_2	10.2683	3.5525	0.2689	0.4039	106.5047	108.0047	107.3005	106.0030
		θ	0.3303	0.0398						
v	EWD	λ_3	39.2764	21.1171	0.1479	0.9810	141.3349	143.0492	141.9401	140.6710
		θ	0.2623	0.0283						

V = (40.1, 59.4, 71.2, 166.5, 204.7, 229.7, 308.3),
U= (3.1, 8.3, 8.9, 9.0, 13.6, 14.9, 16.1),
W= (0.6, 13.4, 15.2, 19.9, 25.0, 30.2, 32.8, 44.4),
R_2 = (0, 0, 0, 0, 0, 0, 0, 1, 0), R_1 =(0, 0, 0, 0, 0, 0, 0, 0, 1), R_3=(0, 0, 0, 0, 0, 0, 0, 2).

Table 7 discusses the ML and Bayesian estimates of the EWD parameters based on the GPHC sample of the SSM. Figure 11 displays the posterior distribution's trace and

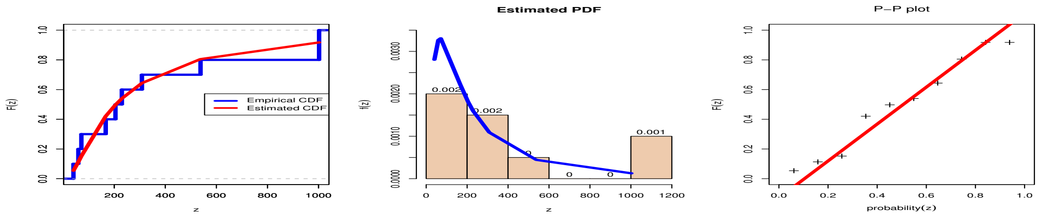

Figure 5: Empirical CDF, histogram, and P-P plots for the EWD for Data set 3

Table 6: ML and Bayesian estimates for reliability in SSM based on complete sample for EWD and EED

		ML				Bayesian			
		Estimates	SE	Lower	Upper	Estimates	SE	Lower	Upper
EWD	λ_1	7.6867	2.5593	2.6704	12.7030	2.5194	0.7976	1.0926	4.0157
	λ_2	9.1172	2.8817	3.4691	14.7654	3.4328	1.1334	1.4230	5.6384
	λ_3	75.7190	34.8958	7.3232	144.1147	72.1292	25.4282	23.7217	122.9561
	θ	0.3036	0.0204	0.2636	0.3437	0.0620	0.0064	0.0526	0.0726
	Υ	0.3744				0.3910			
EED	λ_1	0.6170	0.2216	0.1827	1.0513	0.6506	0.1647	0.2896	1.0329
	λ_2	0.5031	0.1781	0.1540	0.8523	0.5416	0.1716	0.2189	0.8249
	λ_3	3.6845	1.7466	0.2612	7.1077	4.0167	1.6591	1.0943	7.0266
	Υ	0.3445				0.3503			

normal curve for the SSM estimation for EWD based on the GPHC. Figure 12, which displays the pairwise correlation between parameters in the top plot, correlation coefficients in the bottom plot, and the marginal frequency distribution for each parameter on the diagonal, presents the MCMC samples as a pairs plot. Additionally, as shown in Figure 13, convergence starts at 2000 iterations or less for the reliability estimate of the SSM for EWD based on the censored sample.

Table 7: ML and Bayesian estimates for reliability of SSM

		ML				Bayesian			
τ_1, τ_2, τ_3		Estimates	SE	Lower	Upper	Estimates	SE	Lower	Upper
$20, 50, 500$	λ_1	7.7896	2.8023	2.2970	13.2821	8.1878	2.7952	3.3206	14.2400
	λ_2	8.7026	3.0212	2.7810	14.6241	9.3778	2.9498	3.9458	15.1873
	λ_3	68.6337	39.8506	1.4735	146.7409	85.1238	38.3644	15.9283	172.4221
	θ	0.3108	0.0281	0.2557	0.3660	0.3155	0.0250	0.2681	0.3648
	Υ	0.4254				0.4425			
$25, 60, 550$	λ_1	7.8717	2.8090	2.3660	13.3773	8.3905	2.7092	2.7466	14.6373
	λ_2	8.6463	2.9969	2.7725	14.5202	8.9148	2.8299	3.9046	14.4624
	λ_3	74.3011	41.5613	2.1590	155.7611	84.1920	36.4978	21.1738	154.9815
	θ	0.3156	0.0263	0.2641	0.3670	0.3180	0.0230	0.2749	0.3650
	Υ	0.4282				0.4293			

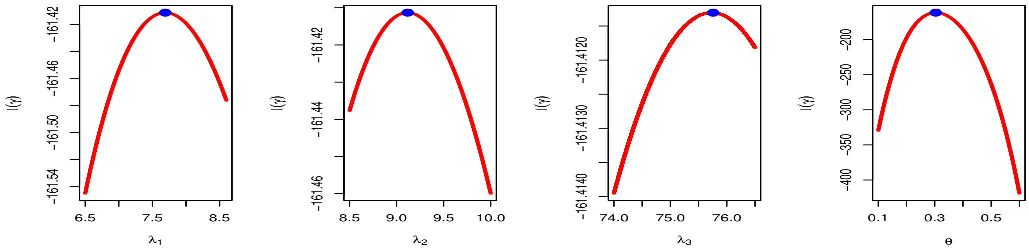

Figure 6: Profile likelihood of EWD parameters

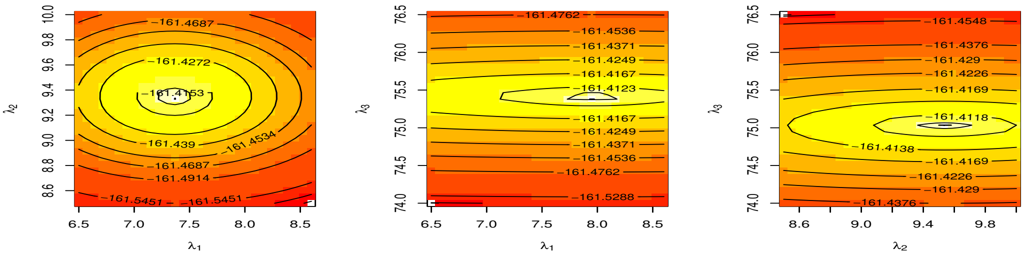

Figure 7: Count-our plots of log-likelihood values with EWD parameters

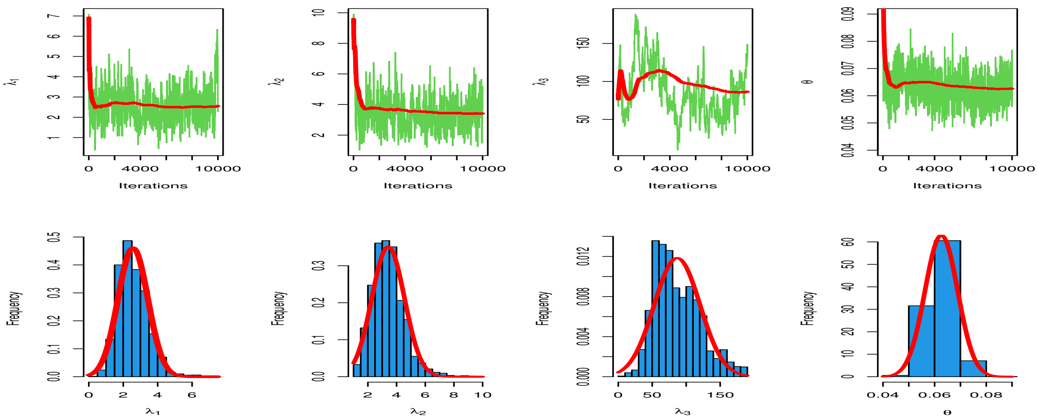

Figure 8: Trace and normal curve of posterior distribution for MCMC estimation of SSM for EWD based on complete sample

7. Summary and Conclusion

In this paper, it is explained how to draw the statistical conclusion that $\Upsilon = P(U < W < V)$ for a component with a strength that is independent of opposite lower and upper bound stresses when the stresses and strength both follow EWD. We presume that the random variables for stresses and strength are both independent and have an EWD with a shared scale parameter. Due to ML and Bayesian methods, various point and interval es-

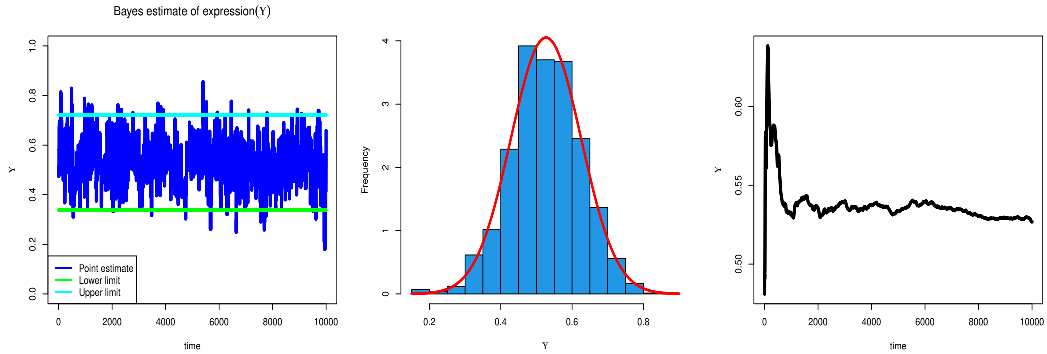

Figure 9: Trace and normal curve of the posterior distribution for MCMC estimation of SSM for EWD based on complete sample

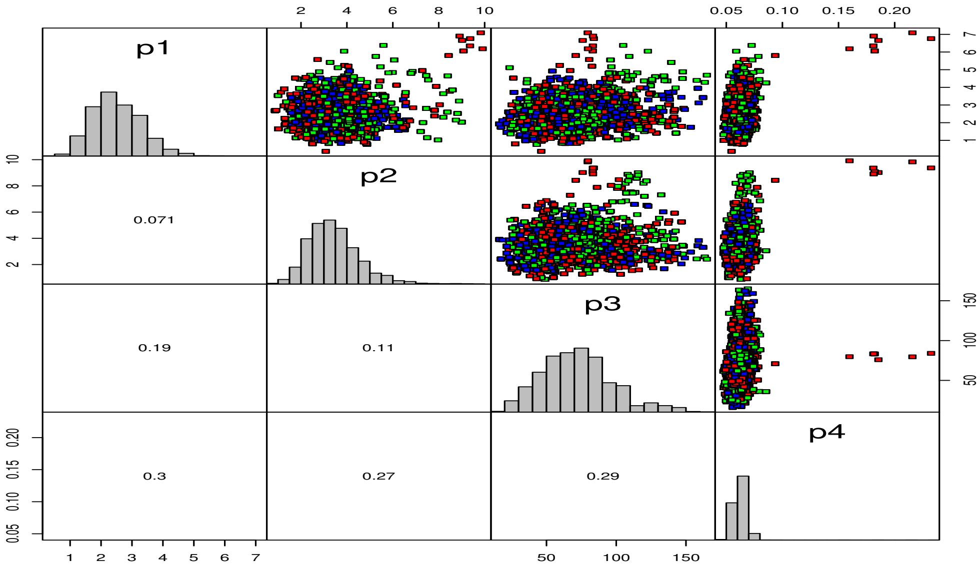

Figure 10: Pairs plot of the MCMC samples for parameter estimates of SSM for EWD based on complete sample

timates for the reliability model Υ are derived using a GPHC design. The MCMC method and the MH algorithm, which are both based on the SELF and LINEX loss functions and are all carried out in the context of informative priors, both produce Bayesian estimators. Asymptotic distribution theory and the construction of Bayes credible intervals are used to derive CIs. Boot-T is preferable to Boot-P, according to discussions on bootstrap CIs. In order to compare the usefulness of the suggested estimates using several metrics, including average values, mean squared error, and length of CIs, the Monte Carlo simulation is carried out. The study's findings show that, for four parameters and SSM based on the GPHC scheme, the Bayes estimates produce lower MSE. For illustrative reasons, an actual data set that has been gradually suppressed is given. This study's main limitations involve using ML estimation and MCMC techniques for $\Upsilon = P(U < W < V)$ computation.

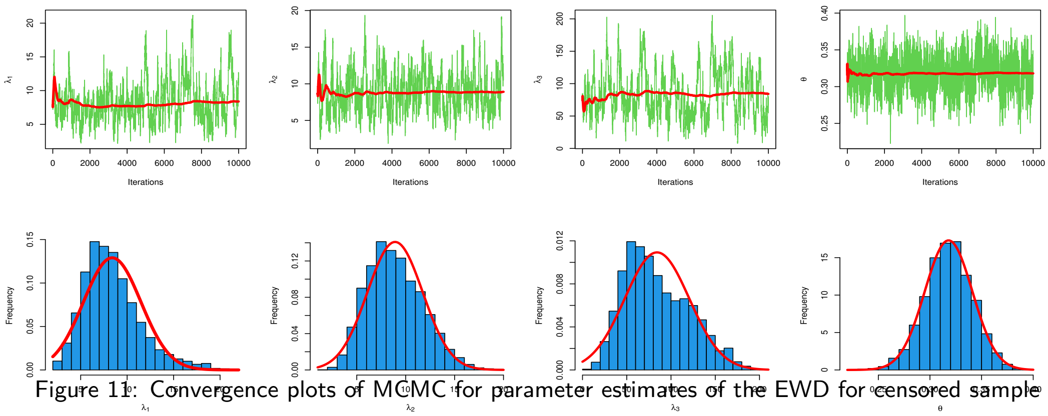

Figure 11: Convergence plots of MCMC for parameter estimates of the EWD for censored sample

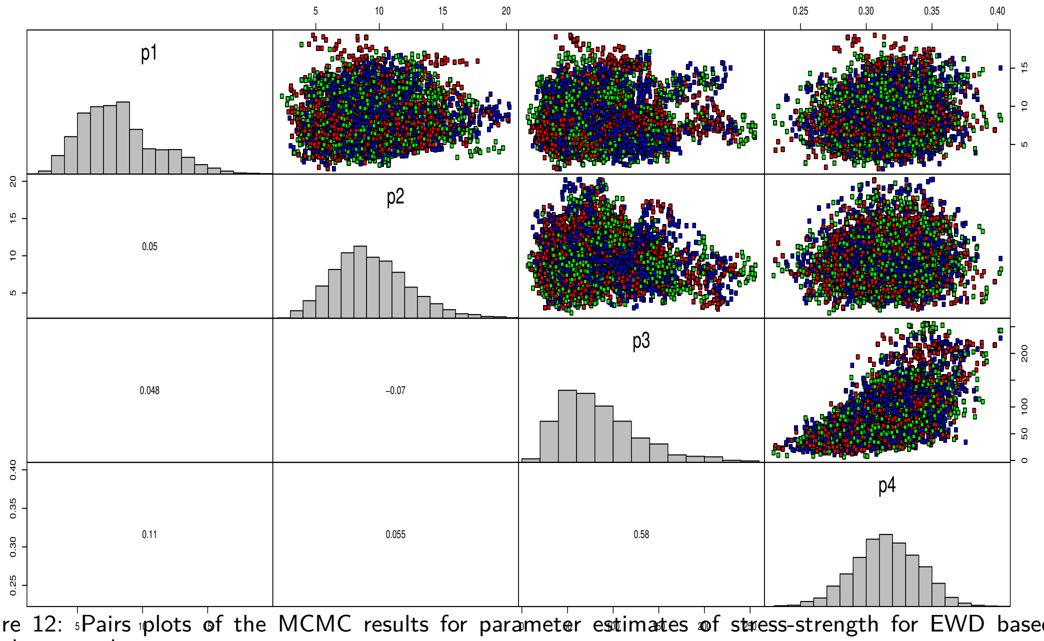

Figure 12: Pairs plots of the MCMC results for parameter estimates of stress-strength for EWD based on complete sample

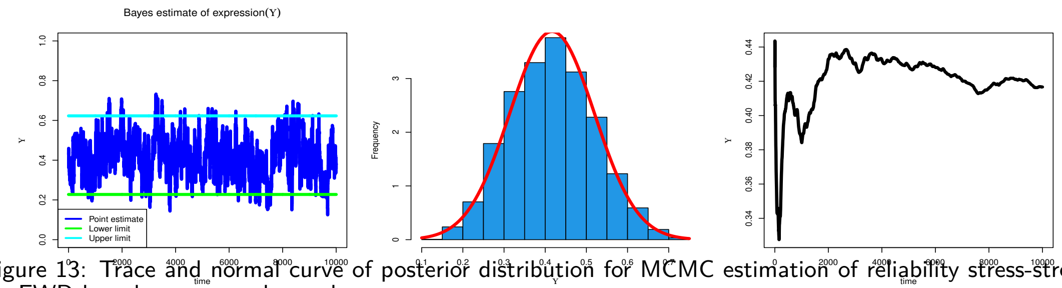

Figure 13: Trace and normal curve of posterior distribution for MCMC estimation of reliability stress-strength for EWD based on censored sample

Future work might consider (1) Tierney-Kadane approximations instead of MCMC, and (2) using the maximum product spacing estimation method as a substitute for the ML estimation method.

Data Availability Statement: The data used to support the findings of this study are included within the article.

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