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Schrödinger Equation and GUP of Attenuated Eigenfunction

A.S. ABDEL-RAHMAN[1,*] AND YOUSSEF A. SABRY[2,*]

[1]*Physics department, Faculty of Science, Cairo University, Giza, Egypt*
[2]*Abul-Houl Academy, College of Mathematics, Giza, Egypt*

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String theory, general relativity, quantum gravity, and the study of black holes all suggest that there could be a minimum observable length of the order of Planck's length. This hint, together with others, led to the Generalized Uncertainty Principle (GUP), which is presented in numerous publications. The Schrödinger equation, which forms the basis of quantum mechanics, does not explicitly demonstrate this principle, but it is often used to solve problems without revealing the full range of possible answers. In this study, a particle in a one-dimensional box, one of the well-known quantum problems, was used to illustrate some implications of the GUP in terms of quantum physics. The study proposes a notation derived from an attenuated wavefunction that describes both the particle dimension and the wave nature of the particle in terms of a minimal length, to be introduced into the Schrödinger equation. Apart from Hawking radiation, this idea can also be used to solve problems in high-energy physics and black holes.

Keywords: Generalized uncertainty, attenuation, minimal length, solution limits

1 INTRODUCTION

The process of reconciling quantum mechanics with the problem of general relativity is one of the tasks of theoretical physics for which a closed-form has not yet been found. Apart from the fact that there is no complete description of quantum gravity, one can predict the existence of a minimal length scale [1]. The existence of a minimal length of measurements should also be expected for

Corresponding author's e-mail: asabry@sci.cu.edu.eg

other reasons [2-4]. Moreover in string theory the associated gravitational effects are assumed to perturb the space-time structure when high energies resolve small distances. In other words: if sufficient amount of mass-energy can be confined to a small region of space, a black hole is created.

According to the theory of quantum gravity, the distance scale must be of the order of Planck's length (L_P). The usual Heisenberg uncertainty principle can be reorganized in terms of minimal uncertainty (Δx_0) in the position (minimal length) as follows:

$$\Delta x \geq \frac{\hbar}{2\Delta p} + \frac{1}{\hbar}\eta L_P^2 \Delta p + ... \tag{1}$$

which is known as the Generalized Uncertainty Principle (GUP). Where $\hbar$ is the reduced Blank's constant ($h/2\pi$), Δx, and Δp are the uncertainties of the position and momentum, respectively. The constant η is independent of the uncertainty quantities but can depend on both the position and the momentum [5-9].

The string theory arguments show the minimal length in the uncertainty as:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} + \eta \hbar L_P^2 \left(\Delta p\right)^2 + \beta \hbar \left(\Delta x\right)^2 + \gamma \hbar \tag{2}$$

Quantum mechanics mainly describes wave propagation, and the fundamental eigenfunction Ψ has the form:

$$\psi = e^{\frac{i}{\hbar}(\mathrm{p}\cdot\mathrm{x} - Et)} \tag{3}$$

where E is the energy eigenvalue, t is the time, and i is the imaginary unit. The momentum vector p can be defined from the differentiation of the eigenfunction to the displacement vector x. The total energy remains the same in classical and quantum mechanical systems, although they differ in their concepts. Schrödinger equation and the Hermit operator describe the summation of momentum operator and potential terms as follows:

$$\hat{H}\psi = V\psi + \frac{\hat{p}^2}{2m}\psi = V\psi - \frac{\hbar^2}{2m}\nabla^2\psi = E\psi \tag{4}$$

The aim of this work is to reconcile of the GUP and the attenuated wave eigenfunction, and to generalize the quantum Schrödinger equation to accept the existence of particle dimensions as well as their wave nature for minimal length problems.

2 METHODOLOGY

It is assumed that all particles, regardless of their mass or size, have their own de Broglie wave, in which the speed of the wave and the particle are the same. The wave is subjected to attenuation (damping) as it propagates. This attenuation is proportional to the displacement x as the wave propagates in the material, while its intensity obeys the inverse square law. Hence, the amplitude of the wave function is inversely proportional to the displacement and can be approximately considered as exponential decay. The eigenfunction can be written as:

$$\psi = e^{-\alpha x+\frac{i}{\hbar}(p\cdot x-Et)} \tag{5}$$

where α is the attenuation coefficient (it is generally space-dependant, but for the problem under-discussion it is constant). To find the momentum, one can apply the differentiation of the eigenfunction to the displacement, which is then as follows:

$$\hat{p}^2 = -\hbar^2\left(\alpha^2 + 2\alpha\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\right) = -\hbar^2\left(\alpha^2 + 2\alpha\nabla + \nabla^2\right) \tag{6}$$

A similar equation was also obtained from the GUP in [3]. The momentum operator consists of three terms: The ∇^2 stands for standard quantum mechanics where the eigenfunctions are waveforms, the α^2 term can be interpreted by classical mechanics where this term alone is not an operator, and the ∇ stands for the intermediate region where quantum mechanics and classical mechanics work; this region is expected and explained in [10-14]. And it is useful for the study of vibrational spectroscopy [15-19] in molecules [20-25], and internal friction [26-31].

Since the wave function cannot be determined in the classical view, Eq. 4 can be approximated according to the new definition of the momentum operator equation 6 as follows:

$$V\psi - \frac{\hbar^2}{2m}2\alpha\nabla\psi - \frac{\hbar^2}{2m}\nabla^2\psi = E\psi \tag{7}$$

This equation consists of both quantum and intermediate regions. This equation will be called the extended Schrödinger equation. Now, the coefficient α should be solved to rewrite the Schrödinger equation in its new form.

3 RESULTS AND DISCUSSION

Following the approach outlined above the well-defined quantum problem can be studied to show Planck's length effect and its term in the extended Schrödinger equation.

3.1 Quantum solution

The current work focuses on the problem of the potential well (1D particle in a box) and can be extended to other problems. The potential of the particle in a 1D-box can be described as follows:

$$V = \begin{cases} 0 & 0 \le x \le L \\ \infty & otherwise \end{cases} \tag{8}$$

where L is the width of the barrier, the quantum solution to this problem is:

$$\begin{aligned} \psi_n &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \\ E_n &= \frac{n^2 \pi^2 \hbar^2}{2mL^2} \end{aligned} \tag{9}$$

where n (=1, 2, 3, …) is an integer representing the energy level. It can be seen that the particle in this box can only have one of the energy levels, namely the lowest level that is not zero, with a small kinetic energy assigned to the particle at this level. In this case, the uncertainty can be expressed as follows:

$$\Delta x \cdot \Delta p = \frac{\hbar}{2} \sqrt{\frac{n^2 \pi^2}{3} - 2} \tag{10}$$

The application of the GUP equation 1 shows the following relationship between the Plank's minimal length and the barrier width:

$$L_P^2 \le \frac{L^2}{2n^2 \pi^2} \left(\sqrt{\frac{n^2 \pi^2}{3} - 2} - 1 \right) \tag{11}$$

The first energy level shows a very small minimal length ($L_p \approx L/12$) compared to the next successive levels, these levels can estimate the minimal length as:

$$L_P \leq \frac{L}{0.64\text{ n} + 4.6} \quad n \geq 2 \tag{12}$$

3.2 Modified quantum solution

Applying Eq. 7 to find the solution when the material validates the quantum and intermediate regions, the eigenfunction can be written as follows (*b* and *c* are constants):

$$\psi = ce^{bx} \tag{13}$$

And then it is converted into a quadratic equation to find the constant *b* and the solution of this equation shows the eigenfunction and eigenstates as:

$$\begin{aligned} \psi_n &= A^{-1} e^{-\alpha x} \sin\left(\frac{n\pi}{L} x\right) \\ A &= \sqrt{\left(e^{-2\alpha L} - 1\right)\left(\frac{\alpha L^2}{4\alpha^2 L^2 + 4n^2\pi^2} - \frac{1}{4\alpha}\right)} \\ E_n &= \frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L^2} + \alpha^2\right) \end{aligned} \tag{14}$$

To obtain the uncertainty of position and momentum, one can determine $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$. They were found as:

$$\begin{aligned} \langle x \rangle &= \frac{1}{A^2}\left\{ e^{-2\alpha L}\left[-\frac{L}{4\alpha} + \frac{4\alpha^3 L^5 + 4n^2\pi^2\alpha L^3}{\left(4\alpha^2 L^2 + 4n^2\pi^2\right)^2}\right] + \left(1 - e^{-2\alpha L}\right)\left[\frac{1}{8\alpha^2} - \frac{2\alpha^2 L^4 - 2n^2\pi^2 L^2}{\left(4\alpha^2 L^2 + 4n^2\pi^2\right)^2}\right]\right\} \\ \langle x^2 \rangle &= \frac{1}{A^2}\left\{ e^{-2\alpha L}\left[\left(-\frac{L^2}{4\alpha} - \frac{L}{4\alpha^2}\right) + \frac{\alpha L^4}{4\left(\alpha^2 L^2 + n^2\pi^2\right)} + \frac{\alpha^2 L^5 - n^2\pi^2 L^3}{4\left(\alpha^2 L^2 + n^2\pi^2\right)^2}\right] + \left(1 - e^{-2\alpha L}\right)\left[\frac{1}{8\alpha^3} - \frac{\alpha^3 L^6 - 3n^2\pi^2\alpha L^4}{8\left(\alpha^2 L^2 + n^2\pi^2\right)^3}\right]\right\} \\ \langle p \rangle &= i\frac{\hbar n\pi}{2A^2\left(2\alpha^2 L^2 + n^2\pi^2\right)}\left\{\left(1 - e^{-2\alpha L}\right)\left[\frac{n\pi}{2} + \frac{\alpha^2 n\pi L^2}{2\left(\alpha^2 L^2 + n^2\pi^2\right)}\right]\right\} \\ \langle p^2 \rangle &= \hbar^2\left\{\alpha^2 + \frac{n^2\pi^2}{L^2}\right\} \end{aligned} \tag{15}$$

The terms Δx and $\Delta p/\hbar$ were found by approximation to the values $\sqrt{3}/2\alpha$ and $\sqrt{2}\alpha$, respectively. Both terms are *L*- and *n*-dependent, and therefore the uncertainty equation according to this eigenfunction (Eq. 14) and the momentum operator (Eq. 7) can be written as follows:

$$\alpha^2 \leq \frac{\sqrt{6}-1}{4L_P^2} \tag{16}$$

The minimal length therefore can be estimated to be 0.6/α and then the extended Schrödinger equation can be rewritten as follows according to the Generalized Uncertainty minimal length:

$$\hat{H}\psi = V\psi - \frac{\hbar^2}{2mL_P}1.2\nabla\psi - \frac{\hbar^2}{2m}\nabla^2\psi = E\psi \tag{17}$$

This equation describes the mechanics in the neighbour of Plank's length (i.e., high-energy physics and black matter). The attenuation coefficient may be imaginary, which will be discussed in the next section.

3.3 Solution limit

Quantum mechanics treats with particles as a matter waves associated with a point mass and neglects the particle dimensions. The 1D particle in a box quantum solution cannot consider a particle associated with a zero energy eigenvalue, but this can be considered when particle size is taken into account. If the barrier width *L*, which can be considered as the range of interaction, is narrow as it encloses the particle and touches its ends; the barrier width *L* is equal to the particle diameter *D*, the particle should be restricted in its position and unable to hold any amount of energy (appears fixed).

For the first energy level E_1, a particle with zero energy can be interpreted in Eq. 14, since the bracket term vanishes. The value of the attenuation coefficient α is:

$$\alpha = i\frac{\pi}{D} \tag{18}$$

This can be explained by the fact that the actual particle size is *D*/π while the residual space of the particle is its field region and is kept free of others. The eigenfunction can be rewritten as:

$$\psi = e^{\frac{i}{\hbar}\left(\left[\breve{p} - \frac{\pi\hbar}{D}\right]\breve{x} - Et\right)} \tag{19}$$

Since the attenuation coefficient is imaginary, the extended Schrödinger equation is written in its complex form as:

$$V\psi + \frac{i\pi\hbar^2}{2mD}\nabla\psi - \frac{\hbar^2}{2m}\nabla^2\psi = E\psi \tag{20}$$

The wavefunction is completely harmonic and has a complex solution in which both the real and imaginary parts are harmonic, and which satisfies the real and imaginary parts of the complex Schrödinger equation Eq. 20. This result is in good agreement with induced nuclear fission. The neutron-nuclei interaction can be assumed to be a potential well problem, in which no Coulomb potentials are present. Neutrons enter fission only when their energy is >1 MeV [32, 33], which corresponds to a momentum of 2.32×10^{-20} Kg.m/s, and the value of the interaction range D (Eq. 16 and Eq. 9) is 4.55 fm. This value is too close to the limit of the potential barrier width $L\approx D$ (the neutron diameter 1.6 fm [34, 35]) divided by π which is 5.03 fm. At this value, the interaction is classical, and above this energy, the scattering cross section increases (similar to quantum wave scattering) and the probability of capture is lower.

One can consider the full form of the modified Schrödinger equation as shown in Eq. 7, where the coefficient α can be restricted to these values for quantum mechanical solutions:

$$-\frac{\pi^2}{D^2} \le \alpha^2 \le \frac{\sqrt{6}-1}{4L_P^2} \tag{21}$$

4 CONCLUSION

The minimal length, expected for many reasons in quantum gravity, and string theory, was found to be of the order of Planck's length, but its appearance in the quantum solutions was not drawn while it appears in the Generalized Uncertainty Principle. The well-known quantum problem; a particle in a one-dimensional box was discussed in terms of the attenuated wavefunction. It concludes with the addition of some terms to the Schrödinger equation to also describe the particle dimension as well as its wave nature according to a minimal length. The solution to the problem shows the limits to which quantum mechanics can still be applied; this result is confirmed by the neutron-induced fission energy. The concept obtained can be used to explain problems with black holes and high-energy physics.

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DATA AVAILABILITY STATEMENT

The datasets calculated or discussed in this work are available from the corresponding author upon a reasonable request.

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