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Kepler's problem of a two-body system perturbed by a third body

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Abstract One of the most important problems in basic physics and astronomy is studying the motion of planets, satellites, and other celestial bodies. The solution to the two-body problem enables astronomers to predict the orbits of the Moon, satellites, and spaceships around the Earth. The general analytic solution for the three-body problem stands unsolved except in some special cases. This reduces the problem to a two-body problem. In this work, the authors present a closed-form approach to the three-body problem theoretically and numerically based on particle–particle vector analysis. The theoretical approach, which is based on the real Moon–Sun–Earth problem information, illustrates the perturbation of the Moon in the Sun–Earth problem and shows an expected orbital motion with a perturbation in the Sun–Earth orbit due to the revolution of the Moon. The numerical investigation uses the same information to study the same problem and calculate the angular momentums of each pair of objects. The two solutions show good agreement with the well-known Earth-Moon and Sun–Earth momentums. The Moon–Sun orbit is close to an elliptic shape with angular momentum of about 3.27×10^{38} J.s. This approach is the key to future studies for n-body problem solutions.

1 Introduction

In classical mechanics, the two-body problem is used to predict the motion of two massive objects that are abstractly viewed as point masses. The problem assumes that the two objects interact only gravitationally with one another; all other forces are ignored.

We consider two bodies of masses m_1 and m_2 with position vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, respectively. Let $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ be the vector positions of the two bodies and m_1 and m_2 be their masses. The goal is to determine the trajectories $\boldsymbol{r_1}(t)$ and $\boldsymbol{r_2}(t)$ for all time t, given the initial positions $\boldsymbol{r_1}(t = 0)$ and $\boldsymbol{r_2}(t = 0)$ and the initial velocities $\dot{\boldsymbol{r}}_{\boldsymbol{1}}(t = 0)$ and $\dot{\boldsymbol{r}}_2(t = 0)$.

Newton's second law states that:

$$\mathbf{F_{12}}(\boldsymbol{r_1}) = m_1\ddot{\boldsymbol{r}}_{\boldsymbol{1}} \tag{1}$$

$$\mathbf{F_{21}}(\boldsymbol{r_2}) = m_2\ddot{\boldsymbol{r}}_2 \tag{2}$$

where $\boldsymbol{F_{12}}$ is the force on m_1 due to its interactions with m_2 and $\boldsymbol{F_{21}}$ is the force on m_2 due to its interactions with m_1. The single or double dots on top of the $\boldsymbol{r}$ position vectors; which are the first or second derivatives with respect to time, represent their velocity or acceleration vectors, respectively.

If $\boldsymbol{r_{12}}$ is the vector distance between the two bodies, its time derivatives may be used to rewrite Eqs. 1 and 2 together as:

$$\ddot{\boldsymbol{r}}_{12} = \ddot{\boldsymbol{r}}_2 - \ddot{\boldsymbol{r}}_{\boldsymbol{1}} = \frac{\mathbf{F_{21}}}{m_2} - \frac{\mathbf{F}_{12}}{m_1} \tag{3}$$

The center of mass vector $\boldsymbol{r_{cm}}$ (where all body masses have an effect) may be introduced from its definition:

$$\sum_{i=1}^{n} m_i(\boldsymbol{r}_i - \boldsymbol{r}_{cm}) = 0 \tag{4}$$

and written generally as:

$$\boldsymbol{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\boldsymbol{r}_i}{\sum_{i=1}^{n} m_i} \tag{5}$$

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Both $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are vector functions in $\boldsymbol{r_{cm}}$ and $\boldsymbol{r_{12}}$. They are deduced from the following matrix form:

$$\begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix} = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_{cm} \\ \boldsymbol{r}_{12} \end{pmatrix} \tag{6}$$

The χ_{ij} values can be calculated from the two independent equations $\boldsymbol{r_2} - \boldsymbol{r_1} = \boldsymbol{r_{12}}$ and $m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} = (m_1 + m_2)\boldsymbol{r_{cm}}$, where the first equation can be written as:

$$(\chi_{21} - \chi_{11})\boldsymbol{r}_{cm} = (1 - \chi_{22} + \chi_{12})\boldsymbol{r}_{12} \tag{7}$$

Since the two vectors $\boldsymbol{r_{cm}}$ and $\boldsymbol{r_{12}}$ are independent, the multiplied scalar terms in each should vanish. In this methodology, the χ_{ij} values can be set in one matrix as follows:

$$\begin{pmatrix} 0 \\ 1 \\ m_1 + m_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ m_1 & m_2 & 0 & 0 \\ 0 & 0 & m_1 & m_2 \end{pmatrix} \begin{pmatrix} \chi_{11} \\ \chi_{21} \\ \chi_{12} \\ \chi_{22} \end{pmatrix} \tag{8}$$

By inverting the central matrix and multiplying it by the left-hand matrix, the χ_{ij} values will be determined, and hence the position vectors of the two-bodies are written as follows:

$$\begin{aligned} \boldsymbol{r}_1(t) &= \boldsymbol{r}_{cm} - \frac{m_2}{m_1 + m_2}\boldsymbol{r}_{12} \\ \boldsymbol{r}_2(t) &= \boldsymbol{r}_{cm} + \frac{m_1}{m_1 + m_2}\boldsymbol{r}_{12} \end{aligned} \tag{9}$$

from which we can deduce the equations of motion for the two-body problem. According to Eqs. 9, the two-body problem is now reduced to a one-body problem, and the two trajectories $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ for all time t can be determined. The relative distance vector $\boldsymbol{r_{12}}$ can be calculated due to the potential $V(\boldsymbol{r_{12}})$ between the two bodies.

Cowell's method [1] in celestial mechanics is the simplest way to solve the perturbed two-body problem. This approach is based on integrating directly the Newtonian equations of motion in rectangular coordinates. If the distance between the two bodies, which are thought of as point masses, becomes zero, the analytical solution for these differential equations is singular. The numerical solution loses accuracy also, even if the bodies are considered to have a definite size.

Regularization is the process of eliminating singularities from the equations of motion by carefully choosing variables in order to get regular differential equations [2]. Levi–Civita [3] and Sundman [4, 5] conducted the first essential lines of research on the regularization of the three-body problem.

The generalization of the special perturbation, in the case of a perturbing force, is entirely or partially derivable from a potential [6]. Generally, the three-body problem solutions are more accurate than two-body problem perturbed by third one. Based on this, the three-body problem will be considered analytically and numerically in this work.

Mathematicians have been drawn to the three-body problem for ages because of its enigmatic nature and simplicity. The topic has been tackled by mathematic giants such as Euler, Lagrange, Laplace, Jacobi, Le Verrier, Hamilton, Poincaré, and Birkhoff, who have all made significant contributions. Szebehely [7] and Marchal [8] both offer scholarly analyses of the relevant literature and derivations of the key findings. The three-body problem remains mysterious until now, despite the fact that a great deal has previously been discovered. However, new findings and renewed interest in the subject have resulted from recent advancements in nonlinear dynamics and the stimulation of additional solar system data [9].

There is no general closed-form solution to the three-body problem [10] (or a more general one for the n-body problem). This is also proved by Bruns [11] in 1887, who stated that there is no closed-form solution for the n-body problem, $n \geq 3$. The work of Euler [12, 13] in 1740 illustrated a family of solutions for many bodies orbiting around a common center of mass. Also, Poincaré [14] established the existence of an infinite number of periodic solutions [15] to the restricted three-body problem in 1906, together with techniques for continuing these solutions into the general three-body problem.

Lagrange [16] in 1867 found a solution for the three-body problem, which forms an equilateral triangle [17]. Hénon and Broucke in the 1970's each found a family of solutions [18] (the Broucke–Henon–Hadjidemetriou family) involving two masses bouncing back and forth in the center of a third body's orbit. Moore [19] in the 1990's discovered a stable figure-eight orbit of three equal masses with a zero-angular-momentum solution; this solution was also proved by Chenciner and Montgomery [20, 21]. Šuvakov and Dmitrašinović [18] also discovered thirteen new families of solutions for the zero-angular-momentum equal-mass three-body problem [22], and Hudomal [23] discovered fourteen families for the same problem.

Stone and Leigh [24] stated that the chaotic motion of the system happens because its state seems to get randomly shuffled over time. The motion is perfectly determined between one instant and the next. But it can be thought of as approximately random over long intervals, such as a pseudo-random system that will, over time, explore all possible configurations consistent with some basic properties like the energy and angular momentum of the system. The system explores what we call "phase space", a space of possible arrangements of position and velocity.

Sundman found a solution to the general three-body problem as a converging infinite series that was added together to solve the orbit calculation [25, 26]. Because the series converged, successive terms diminished to effectively nothing, so in principle, the equation could be written out on paper. However, the convergence of Sundman's series is so slow that it would take billions of terms to converge for typical calculations in celestial mechanics.

The closed form of the three-body or n-body problem is still unsolved, but the three-body problem is particularly important for astronomers and physicists studying the motion of the Moon–Sun–Earth system. The three-body problem is also solvable in restricted cases, such as a body of negligible mass (the "planetoid") moving under the influence of two massive bodies.

In this work, an approach to the three-body system's general solution will be presented and discussed. The Moon–Sun–Earth problem was restudied according to the Moon's influence on the Sun–Earth orbit as a third-body perturbation. The three-body problem solution, which is deduced in this work, was applied to the same system to show how it explained the orbits numerically. The application of this solution is important for the motion of planets, rotation, vibration [27–31], etc., of molecules [32–37], internal friction [38–44], and other point mass systems in physics [45–54].

The most important application of this work is in astronomy, especially in satellite orbit design. Traditional satellite engineering designs the orbit according to the rotation of the body around the Earth. Due to the lack of many-body system solutions, engineers use a two-body equation in the main problem. In the real environment, the Earth's Moon (and other celestial bodies) affects the satellite orbit, and their effect will be clear after a number of satellite revolutions as the orbit becomes narrower. So the satellite orbit should be corrected (the orbit recovery maneuver) by using propulsion engines that consume fuel. As the fuel runs out, the satellite becomes out of control and hence falls down into the Earth's atmosphere and is destroyed. The solution to the three-body problem can improve the orbit properties where some meters can be added or subtracted from the two-body orbit axes length. This will make the orbit more stable, and hence, the satellite lifetime increases which decreases the expenses of satellites.

2 Methodology

To solve the three-body system, we need to find the $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r_3}$ vectors (at any time t) in terms of the center of mass and relative distance vectors between every two masses ($\boldsymbol{r_{cm}}$, $\boldsymbol{r_{12}}$, $\boldsymbol{r_{23}}$, and $\boldsymbol{r_{31}}$). So twelve unknown variables are expected (four variables in front of each $\boldsymbol{r_{cm}}$, $\boldsymbol{r_{12}}$, $\boldsymbol{r_{23}}$, and $\boldsymbol{r_{31}}$ in three independent equations of the $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r_3}$ vectors). The exact solution requires twelve independent equations. According to the available equations, only three general equations are written ($\boldsymbol{r_3} - \boldsymbol{r_2}$, $\boldsymbol{r_1} - \boldsymbol{r_3}$, and $m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} + m_3\boldsymbol{r_3}$), and the fourth, $\boldsymbol{r_2} - \boldsymbol{r_1}$, will be dropped because it is just a combination of the first two equations.

Due to the leak of enough equations, the solution is hard to find. One can expect the unity of three variables in front of $\boldsymbol{r_{cm}}$ in the three equations (as that obtained in a two-body problem), using the fact of the relative distance vectors as in Fig. 1 which is given by:

$$\boldsymbol{r}_{12}(t) + \boldsymbol{r}_{23}(t) + \boldsymbol{r}_{31}(t) = 0 \tag{10}$$

The vector $\boldsymbol{r_{12}}$ will be cancelled out of the treatment since it can be generated from the two vectors $\boldsymbol{r_{23}}$ and $\boldsymbol{r_{31}}$ in the three equations, and they will be written as:

$$\begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_{cm} \\ \boldsymbol{r}_{cm} \\ \boldsymbol{r}_{cm} \end{pmatrix} + \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \\ \chi_{31} & \chi_{32} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_{23} \\ \boldsymbol{r}_{31} \end{pmatrix} \tag{11}$$

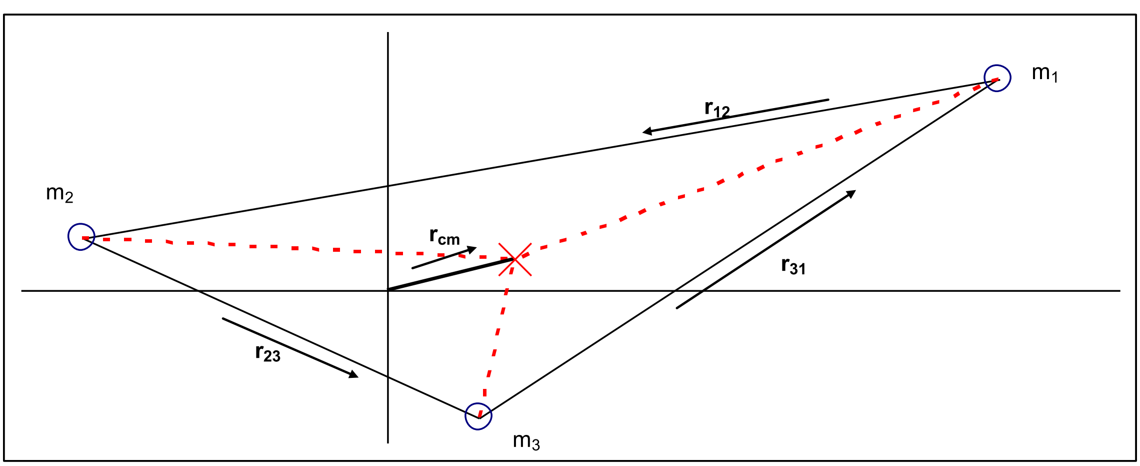

Fig. 1 Three-body system

By using the same methodology in a two-body problem, there are three independent equations ($\boldsymbol{r_3} - \boldsymbol{r_2}$, $\boldsymbol{r_1} - \boldsymbol{r_3}$, and $m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} + m_3\boldsymbol{r_3}$), each of them in a similar form to Eq. (7). There will be six independent equations in six unknowns (χ_{ij}) that can be grouped in the matrix form as follows:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ m_1 & 0 & m_2 & 0 & m_3 & 0 \\ 0 & m_1 & 0 & m_2 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{21} \\ \chi_{22} \\ \chi_{31} \\ \chi_{32} \end{pmatrix} \tag{12}$$

The solution of this matrix gives the following three position vectors to obtain the equations of motion for the three-body problem:

$$\begin{aligned} \boldsymbol{r}_1(t) &= \boldsymbol{r}_{cm} + \frac{m_2}{m_1 + m_2 + m_3}\boldsymbol{r}_{23} + \frac{m_2 + m_3}{m_1 + m_2 + m_3}\boldsymbol{r}_{31} \\ \boldsymbol{r}_2(t) &= \boldsymbol{r}_{cm} - \frac{m_1 + m_3}{m_1 + m_2 + m_3}\boldsymbol{r}_{23} - \frac{m_1}{m_1 + m_2 + m_3}\boldsymbol{r}_{31} \\ \boldsymbol{r}_3(t) &= \boldsymbol{r}_{cm} + \frac{m_2}{m_1 + m_2 + m_3}\boldsymbol{r}_{23} - \frac{m_1}{m_1 + m_2 + m_3}\boldsymbol{r}_{31} \end{aligned} \tag{13}$$

Another methodology can directly find the same solution based on the definition of the center of mass (Eq. 5). The basic vectors $\boldsymbol{r_3}$ and $\boldsymbol{r_2}$ can be written as:

$$\begin{aligned} \boldsymbol{r}_3 &= \boldsymbol{r}_1 - \boldsymbol{r}_{31} \\ \boldsymbol{r}_2 &= \boldsymbol{r}_3 - \boldsymbol{r}_{23} = \boldsymbol{r}_1 - \boldsymbol{r}_{23} - \boldsymbol{r}_{31} \end{aligned} \tag{14}$$

By substituting Eq. 14 in Eq. 5, one can write:

$$\begin{aligned} (m_1 + m_2 + m_3)\boldsymbol{r}_{cm} &= m_1\boldsymbol{r}_1 + m_2(\boldsymbol{r}_1 - \boldsymbol{r}_{23} - \boldsymbol{r}_{31}) + m_3(\boldsymbol{r}_1 - \boldsymbol{r}_{31}) \\ &= (m_1 + m_2 + m_3)\boldsymbol{r}_1 - m_2\boldsymbol{r}_{23} - (m_2 + m_3)\boldsymbol{r}_{31} \end{aligned} \tag{15}$$

The vector $\boldsymbol{r_1}$ can be written as:

$$\boldsymbol{r}_1 = \boldsymbol{r}_{cm} + \frac{m_2}{m_1 + m_2 + m_3}\boldsymbol{r}_{23} + \frac{m_2 + m_3}{m_1 + m_2 + m_3}\boldsymbol{r}_{31} \tag{16}$$

The other vectors ($\boldsymbol{r_2}$ and $\boldsymbol{r_3}$) can be deduced in the same way.

3 Results and discussion

3.1 Gravitational three-body problem

The three-body analytical solution (Eq. 13) can be applied to three particles with mutual gravitational potentials and centripetal forces. The equation of motion for these bodies can be written as:

$$\begin{aligned} m_1\ddot{\boldsymbol{r}}_1 &= \mathbf{F}_{21} + \mathbf{F}_{31} = -G\frac{m_1 m_2}{r_{12}^3}\boldsymbol{r}_{21} + \frac{L_{12}^2}{m_1 r_{12}^4}\boldsymbol{r}_{21} - G\frac{m_1 m_3}{r_{13}^3}\boldsymbol{r}_{31} + \frac{L_{31}^2}{m_1 r_{13}^4}\boldsymbol{r}_{31} \\ m_2\ddot{\boldsymbol{r}}_2 &= \mathbf{F}_{12} + \mathbf{F}_{32} = -G\frac{m_1 m_2}{r_{12}^3}\boldsymbol{r}_{12} + \frac{L_{12}^2}{m_2 r_{12}^4}\boldsymbol{r}_{12} - G\frac{m_2 m_3}{r_{32}^3}\boldsymbol{r}_{32} + \frac{L_{23}^2}{m_2 r_{23}^4}\boldsymbol{r}_{32} \\ m_3\ddot{\boldsymbol{r}}_3 &= \mathbf{F}_{23} + \mathbf{F}_{13} = -G\frac{m_2 m_3}{r_{23}^3}\boldsymbol{r}_{23} + \frac{L_{23}^2}{m_3 r_{23}^4}\boldsymbol{r}_{23} - G\frac{m_1 m_3}{r_{13}^3}\boldsymbol{r}_{13} + \frac{L_{13}^2}{m_3 r_{13}^4}\boldsymbol{r}_{13} \end{aligned} \tag{17}$$

where G is the general gravitational constant and L_{12}, L_{23}, and L_{31} are the angular momenta of the three bodies in terms of the relative distance vectors. These equations are similar to that presented in [55, 56], where they consider the center of mass of the first two bodies and then interacts with the third one. This is not identical to what is presented here. The third body is treated without neglecting its mass and hence the total center of mass ($\boldsymbol{r_{cm}}$) is considered.

Each force between any two bodies has a gravitational potential and a centripetal potential; this is the effective potential of a rotating frame of reference. The centripetal force is related to the angular momentum of rotation; it is conserved in a two-body problem since no external torque is applied. The total energy in the three-body problem is conserved, but generally the angular momentum is not since each of the two rotating bodies is under the external torque of the third one. In our case, the Moon–Sun–Earth problem, the influence of the Moon on the Sun–Earth orbit is so small (clarification is at the end of this section). So the angular momentums L_{23} and L_{31} are identical to the Sun–Earth and Earth–Moon angular momentums in the two-body problem. Since the moon's orbit around the sun is nearly elliptical, the Moon–Sun angular momentum L_{12} should be considered. It can be approximated as a constant because the variation of the rotation angle is very small. This will be verified in the numerical approach section.

The general solution for this case will be written as follows:

$$\begin{aligned}
m_1m_2\ddot{\boldsymbol{r}}_{23} + (m_1m_2 + m_1m_3)\ddot{\boldsymbol{r}}_{31} &= \left(-GM\frac{m_1m_2}{r_{12}^3} + \frac{ML_{12}^2}{m_1r_{12}^4}\right)\boldsymbol{r}_{23} \\
&\quad + \left(-GM\frac{m_1m_2}{r_{12}^3} + \frac{ML_{12}^2}{m_1r_{12}^4} - GM\frac{m_1m_3}{r_{13}^3} + \frac{ML_{31}^2}{m_1r_{13}^4}\right)\boldsymbol{r}_{31} \\
(m_1m_2 + m_2m_3)\ddot{\boldsymbol{r}}_{23} + m_1m_2\ddot{\boldsymbol{r}}_{31} &= \left(-GM\frac{m_1m_2}{r_{12}^3} + \frac{ML_{12}^2}{m_2r_{12}^4} - GM\frac{m_2m_3}{r_{23}^3} + \frac{ML_{23}^2}{m_2r_{23}^4}\right)\boldsymbol{r}_{23} \\
&\quad + \left(-GM\frac{m_1m_2}{r_{12}^3} + \frac{ML_{12}^2}{m_2r_{12}^4}\right)\boldsymbol{r}_{31} \\
m_2m_3\ddot{\boldsymbol{r}}_{23} - m_1m_3\ddot{\boldsymbol{r}}_{31} &= \left(-GM\frac{m_2m_3}{r_{23}^3} + \frac{ML_{23}^2}{m_3r_{23}^4}\right)\boldsymbol{r}_{23} - \left(-GM\frac{m_1m_3}{r_{13}^3} + \frac{ML_{13}^2}{m_3r_{13}^4}\right)\boldsymbol{r}_{31}
\end{aligned} \tag{18}$$

where M is the sum of the system masses. One can rewrite these equations in terms of one acceleration vector to clearly show the direction of each acceleration vector with respect to the particle–particle vector:

$$\begin{aligned}
m_1m_3\ddot{\boldsymbol{r}}_{31} &= \left(-G\left[\frac{m_1m_3(m_1+m_3)}{r_{13}^3} + \frac{m_1m_2m_3}{r_{12}^3}\right] + \frac{L_{31}^2}{r_{31}^4}\left[\frac{m_1}{m_3} + \frac{m_3}{m_1}\right] + \frac{m_3L_{12}^2}{m_1r_{12}^4}\right)\boldsymbol{r}_{31} \\
&\quad + \left(-Gm_1m_2m_3\left[\frac{1}{r_{12}^3} - \frac{1}{r_{23}^3}\right] + \frac{m_3L_{12}^2}{m_1r_{12}^4} - \frac{m_1L_{23}^2}{m_3r_{23}^4}\right)\boldsymbol{r}_{23} \\
m_2m_3\ddot{\boldsymbol{r}}_{23} &= \left(-G\left[\frac{m_2m_3(m_2+m_3)}{r_{23}^3} + \frac{m_1m_2m_3}{r_{12}^3}\right] + \frac{L_{23}^2}{r_{23}^4}\left[\frac{m_2}{m_3} + \frac{m_3}{m_2}\right] + \frac{m_3L_{12}^2}{m_2r_{12}^4}\right)\boldsymbol{r}_{23} \\
&\quad + \left(-Gm_1m_2m_3\left[\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right] + \frac{m_3L_{12}^2}{m_2r_{12}^4} - \frac{m_2L_{13}^2}{m_3r_{13}^4}\right)\boldsymbol{r}_{31}
\end{aligned} \tag{19}$$

It is clear that the particle–particle acceleration vectors have components for each vector, and thus, the solution to this deferential equation is very difficult, except in some special cases.

3.2 Theoretical approach to the Moon–Sun–Earth problem

An important solution for astronomers and physicists is the Moon–Sun–Earth problem; this case was studied as a two-body problem by neglecting the Moon's mass as affected by the forces of the Sun and Earth. The masses were $m_1 = 7.35 \times 10^{22}$ kg, $m_2 = 1.99 \times 10^{30}$ kg, and $m_3 = 5.97 \times 10^{24}$ kg for the Moon, Sun, and Earth, respectively. The gravitational force constant G is taken as 6.67×10^{-11} N m^2/kg^2. The solutions to the two-body problem due to Kepler were (m is the body's mass, which revolves around M body mass):

$$r = \frac{a(1-e^2)}{1-e\cos\theta},\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{L}{mr^2},\ a = \frac{L^2}{(1-e^2)GMm^2},\ T = \sqrt{\frac{4\pi^2a^3}{GM}} \tag{20}$$

In this work, this case will be investigated without neglecting any parties. If the three masses are selected to be the Moon, Sun, and Earth, respectively, one can notice the Moon–Sun and Sun–Earth vector lengths are nearly equal and is very large compared to Moon–Earth vector length ($|\boldsymbol{r}_{12}| \approx |\boldsymbol{r}_{23}| \gg |\boldsymbol{r}_{13}|$). The Earth's mass is too large (nearly one hundred times) compared to that of Moon ($m_3 \gg m_1$); so the $\ddot{\boldsymbol{r}}_{31}$ equation of Eq. 19 could be approximately written as:

$$\ddot{\boldsymbol{r}}_{31} = \left(-G\frac{m_1+m_3}{r_{13}^3} + \frac{L_{31}^2}{r_{13}^4}\left[\frac{1}{m_1^2} + \frac{1}{m_3^2}\right]\right)\boldsymbol{r}_{31} \tag{21}$$

The solution to this differential equation can be written as:

$$\begin{aligned}
r_{31} &= \frac{a_{31}(1-e_{31}^2)}{1-e_{31}\cos\theta_{31}}, &\quad \frac{\mathrm{d}\theta_{31}}{\mathrm{d}t} &= \frac{L_{31}\sqrt{m_1^2+m_3^2}}{r_{31}^2m_1m_3}, \\
L_{31} &= \sqrt{\frac{(1-e_{31}^2)Gm_1^2m_3^2(m_1+m_3)a_{31}}{(m_1^2+m_3^2)}}, &\quad T_{31} &= \sqrt{\frac{4\pi^2a_{31}^3}{G(m_1+m_3)}}
\end{aligned} \tag{22}$$

Table 1 Earth–Moon orbit data [59–61]

Term	Definition	Unit	Value	Error	References
a	Orbital major axis length	m	3.84×10^8 observed		[53–56, 59–61]
e	Eccentricity		5.49×10^{-2} observed		[53–56, 59–61]
L	Angular momentum	J.s	2.900×10^{34} observed		[60, 61]
			2.870×10^{34} Kepler solution	1.0423	
			2.887×10^{34} this work	0.4426	
T	Revolution periodic	s	2.361×10^6 observed		[60]
			2.369×10^6 Kepler solution	0.3692	
			2.355×10^6 this work	0.2430	

By applying these equations with the given values of the orbit's major axis and eccentricity, the angular momentum of the moon and the periodic time of the moon's orbit are calculated. The obtained values are presented in Table 1 beside the absolute error.

The obtained results of this work show more accurate values than those of Kepler for angular momentum and periodic time, which validates the accuracy of this method in calculating the orbit of satellites due to the Satellite–Earth–Moon problem.

So the $\ddot{\boldsymbol{r}}_{\mathbf{23}}$ equation of Eq. 19 could be approximately written as follows:

$$\ddot{\boldsymbol{r}}_{23} = \left(-G\frac{M}{r_{23}^3} + \frac{L_{23}^2}{r_{23}^4}\left[\frac{1}{m_2^2} + \frac{1}{m_3^2}\right]\right)\boldsymbol{r}_{23} - \left(-G\frac{m_1}{r_{31}^3} + \frac{L_{31}^2}{m_3^2 r_{31}^4}\right)\boldsymbol{r}_{31} \tag{23}$$

The Sun–Earth orbit was affected by the presence of the moon in the system. As the length of $\boldsymbol{r}_{\mathbf{31}}$ is smaller than that of $\boldsymbol{r}_{\mathbf{23}}$, one can assume the $\boldsymbol{r}_{\mathbf{31}}$-term in Eq. 23 is a perturbation to the elliptic orbit $\boldsymbol{r}_{\mathbf{23}}$. Since the Earth–Moon orbit plane is inclined by 5.15° to the Sun–Earth orbit plane, the current solution will neglect this orbital inclination and assume Moon–Sun–Earth is a planar problem.

To solve Eq. 23, the perturbation limits only will be considered since the moon's revolution around the earth is nearly circular (the Moon–Earth orbit eccentricity is too small). The Earth's revolution radius around the sun is a harmonic function of the Moon–Earth angle. The extreme points are enough to determine the full revolution function; hence, the maximum and minimum radii can be studied in the following two cases with introducing the unit vector $\hat{\boldsymbol{r}}_{\mathbf{23}}$.

Case 1: When Earth is located between the Sun and the Moon (i.e., $\theta_{31} - \theta_{23} = (2n+1)\pi$, $n = 0,1,2,3,\ldots$), the gravitational force due to the Moon on the Earth reduces the gravitational force due to the Sun on the Earth, and Eq. 23 can be written as:

$$\ddot{\boldsymbol{r}}_{23}\big|_{\theta_{31}-\theta_{23}=(2n+1)\pi} = \left(-G\left(\frac{M}{r_{23}^2} - \frac{m_1}{r_{31}^2}\right) + \frac{L_{23}^2}{r_{23}^3}\left[\frac{1}{m_2^2} + \frac{1}{m_3^2}\right] - \frac{L_{31}^2}{m_3^2 r_{31}^3}\right)\hat{\boldsymbol{r}}_{23} \tag{24}$$

Case 2: When the Moon becomes between the Sun and the Earth (i.e., $\theta_{31} - \theta_{23} = (2n)\pi$), the gravitational forces due to the Sun and the Moon on the Earth become greater, and Eq. 23 will be written as:

$$\ddot{\boldsymbol{r}}_{23}\big|_{\theta_{31}-\theta_{23}=(2n)\pi} = \left(-G\left(\frac{M}{r_{23}^2} + \frac{m_1}{r_{31}^2}\right) + \frac{L_{23}^2}{r_{23}^3}\left[\frac{1}{m_2^2} + \frac{1}{m_3^2}\right] + \frac{L_{31}^2}{m_3^2 r_{31}^3}\right)\hat{\boldsymbol{r}}_{23} \tag{25}$$

The third term in both of Eqs. 24 and 25 is very small compared to the middle term and will be neglected. Rewriting them as:

$$\ddot{\boldsymbol{r}}_{23} = \left(-\frac{GM}{r_{23}^2}\left(1 - \frac{m_1\langle r_{23}^2\rangle}{M\langle r_{31}^2\rangle}\cos(\theta_{31} - \theta_{23})\right) + \frac{L_{23}^2}{r_{23}^3}\left[\frac{1}{m_2^2} + \frac{1}{m_3^2}\right]\right)\hat{\boldsymbol{r}}_{23} \tag{26}$$

The mean values were used instead of the lengths:

$$\begin{aligned}\langle r\rangle &= \int_0^{2\pi} \frac{a\left(1-e^2\right)}{1-e\cos\theta}\mathrm{d}\theta = 2\pi a\sqrt{1-e^2}\\ \langle r^2\rangle &= \int_0^{2\pi} \frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^2}\mathrm{d}\theta = 2\pi a^2\sqrt{1-e^2}\end{aligned} \tag{27}$$

Although the harmonic term in Eq. 26 is time-dependent, its variation is very small and can be considered as a time-independent value. The solution to Eq. 26 will be:

$$\begin{aligned} r_{23} &= \frac{a_{23}\left(1-e_{23}^2\right)}{1-e_{23}\cos\theta_{23}}\cdot\frac{1}{1-\gamma\cos(\theta_{31}-\theta_{23})}\\ L_{23} &= \sqrt{\frac{a_{23}\left(1-e_{23}^2\right)GMm_2^2m_3^2}{\left(m_2^2+m_3^2\right)}}\end{aligned}$$

Table 2 Sun-Earth orbit data [59–61]

Term	Definition	Unit	Value	Error (%)	References
a	Orbital major axis length	m	1.50×10^{11} observed		[59–61]
e	Eccentricity		1.67×10^{-2} observed		[59–61]
L	Angular momentum	J.s	2.660×10^{40} observed		[60, 61]
			2.659×10^{40} Kepler solution	0.0255	
			2.659×10^{40} this work	0.0253	
T	Revolution periodic	s	3.156×10^{7} observed		[60]
			3.156×10^{7} Kepler solution	0.0173	
			3.156×10^{7} this work	0.0152	

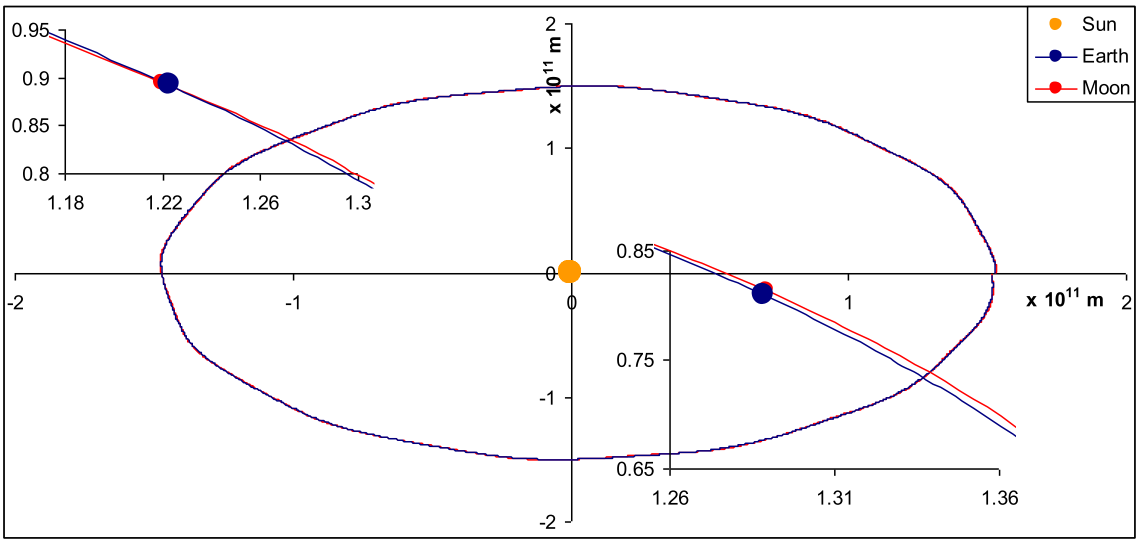

Fig. 2 Moon–Sun–Earth three-body problem

$$\gamma = \frac{m_1 a_{23}^2}{M a_{31}^2}\sqrt{\frac{1-e_{23}^2}{1-e_{31}^2}}$$

$$\frac{d\theta_{23}}{dt} = \frac{L_{23}\sqrt{m_2^2+m_3^2}}{r_{23}^2 m_2 m_3}$$

$$T_{23} = \sqrt{\frac{4\pi^2 a_{23}^3}{GM}} \tag{28}$$

This solution was used for calculating the Sun–Earth orbit characteristics, which are displayed in Table 2.

The approximation in Eq. 21 can show an approximate value for the unknown Moon–Sun angular momentum L_{12} of 3.2749×10^{38} J.s, assuming that its orbit is too close to an ellipse (the orbit is nearly an ellipse where the Earth–Sun distance is one thousand the Earth–Moon distance).

The solution of the Earth–Sun orbit is a perturbed ellipse with a constant γ of about 5.611×10^{-3}. Figure 2 shows the Moon—Sun–Earth orbits, which are in good agreement with [57, 58]. The locations of the Earth and the Moon have also been shown in two different positions (same phases) in the inset figures in Fig. 2.

3.3 Numerical approach to the Moon–Sun–Earth problem

A hand-made VBA application was built to use the Kepler solution for Earth–Moon and Sun–Earth orbits to draw a full system position with a time interval of 863.27 s. The three-body equation (Eq. 19) was considered to estimate the values of angular momenta L_{12}, L_{23}, and L_{31} without any further approximations. The vectors $\boldsymbol{r_{23}}$, $\boldsymbol{r_{13}}$, $\ddot{\boldsymbol{r}}_{31}$, and $\ddot{\boldsymbol{r}}_{23}$ are analyzed in polar coordinates as follows:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} + (r\theta)\hat{\boldsymbol{\theta}} = r\hat{\boldsymbol{r}} + \beta\hat{\boldsymbol{\theta}} \tag{29}$$

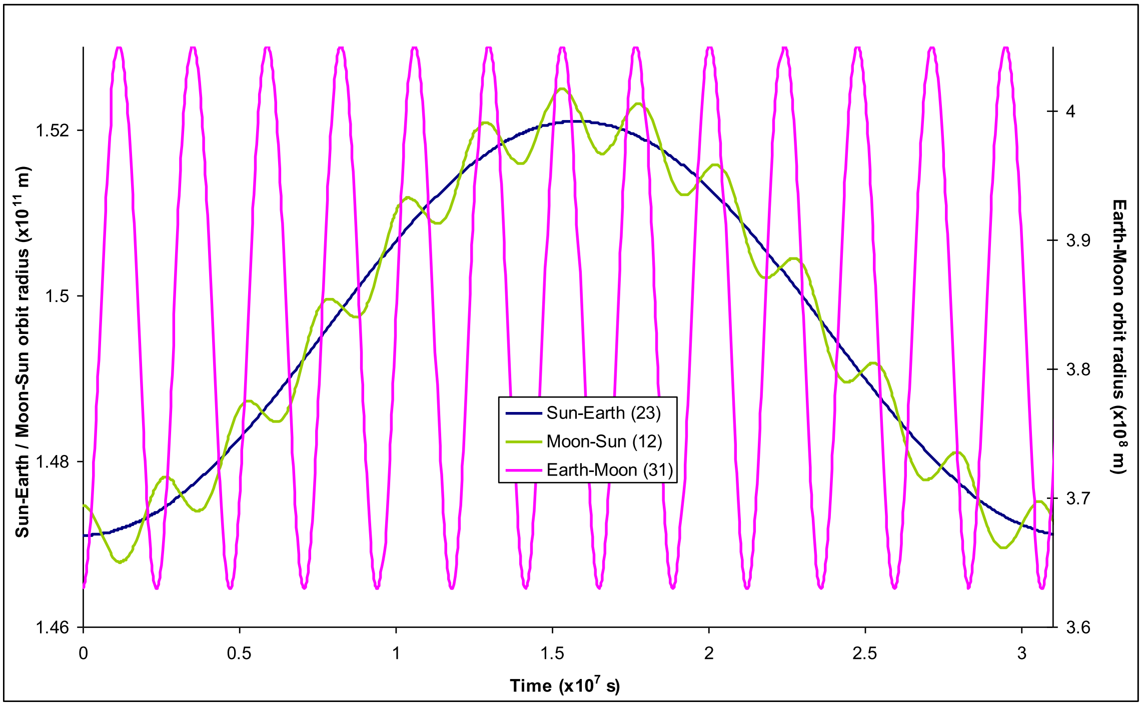

Fig. 3 Moon–Sun–Earth orbital radii

where r represents the radial term and β represents the tangent term of the vector. Based on this, the orbital radii of the system are shown in Fig. 3.

The obtained results for the Earth's tangential speed component around the Sun and for the Moon's tangential speed component around the Earth are about 30 km/s and about 1 km/s, respectively; these values are close to those of the real values.

Equation 19 can be written in terms of four parameters (a_1, a_2, a_3, and a_4) as:

$$\begin{aligned}\ddot{r}_{31} &= a_1 r_{31} + a_2 r_{23}\\ \ddot{\beta}_{31} &= a_1 \beta_{31} + a_2 \beta_{23}\\ \ddot{r}_{23} &= a_3 r_{23} + a_4 r_{31}\\ \ddot{\beta}_{23} &= a_3 \beta_{23} + a_4 \beta_{31}\end{aligned} \tag{30}$$

They can be solved simultaneously to produce these parameters. They will be used in the following equation to get the squares of the system's angular momenta. The solution produces time-dependent momenta; Table 3 shows these results.

$$\begin{aligned}L_{31}^2\left[\frac{1}{m_1^2}+\frac{1}{m_3^2}\right]\frac{1}{r_{31}^4}+L_{12}^2\left[\frac{1}{m_1^2 r_{12}^4}\right] &= a_1 + G\left[\frac{(m_1+m_3)}{r_{13}^3}+\frac{m_2}{r_{12}^3}\right]\\ L_{12}^2\left[\frac{1}{m_1^2 r_{12}^4}\right]-L_{23}^2\left[\frac{1}{m_3^2 r_{23}^4}\right] &= a_2 + G\left[\frac{1}{r_{12}^3}-\frac{1}{r_{23}^3}\right]m_2\\ L_{23}^2\left[\frac{1}{m_2^2}+\frac{1}{m_3^2}\right]\frac{1}{r_{23}^4}+L_{12}^2\left[\frac{1}{m_2^2 r_{12}^4}\right] &= a_3 + G\left[\frac{(m_2+m_3)}{r_{23}^3}+\frac{m_1}{r_{12}^3}\right]\\ L_{12}^2\left[\frac{1}{m_2^2 r_{12}^4}\right]-L_{13}^2\left[\frac{1}{m_3^2 r_{13}^4}\right] &= a_4 + G\left[\frac{1}{r_{12}^3}-\frac{1}{r_{13}^3}\right]m_1\end{aligned} \tag{31}$$

The numerical approaches validate the same results obtained from the approximate theoretical solution.

4 Conclusions

Finding the closed-form solution for the general three-body problem is essential for describing the orbital motions of planets, satellites, and other celestial bodies. The solution based on nonzero-angular momentum was presented for the Moon–Sun–Earth

Table 3 Numerical approach results

Term	Definition	Unit	Value	Tolerance/error (%)
L_{12}	Moon's angular momentum around the Sun	J.s	3.2338×10^{38} Min	
			3.3256×10^{38} Max	2.80×10^{-4} tolerance
			3.2745×10^{38} Avg	1.22×10^{-6} error
			3.2749×10^{38} Cal	
L_{23}	Earth's angular momentum around the Sun	J.s	2.6593×10^{40} Min	
			2.6594×10^{40} Max	3.80×10^{-7} tolerance
			2.6593×10^{40} Avg	2.63×10^{-6} error
			2.660×10^{40} Obs	
L_{31}	Moon's angular momentum around the Earth	J.s	2.8043×10^{34} Min	
			2.9727×10^{34} Max	5.82×10^{-4} tolerance
			2.8922×10^{34} Avg	2.69×10^{-5} error
			2.900×10^{34} Obs	

problem without neglecting any mass. The solution to this problem shows an ordinary Earth-Moon orbit with angular momentum too close to that found in the two-body problem. The Sun–Earth orbit shows small perturbation oscillations due to the effect of the Moon's revolution around the Earth. The Moon–Sun total angular momentum was determined.

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Author contribution The authors confirm contribution to the paper as follows: ASA-R and YAS contributed to the study conception and design; ASA-R collected the data; ASA-R and YAS analyzed and interpreted the results; ASA-R and EMA drafted the manuscript preparation; YAS performed the numerical analysis. All authors reviewed the results and approved the final version of the manuscript.

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Declarations

Conflict of interest Authors declare no conflict of interest.

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