Siméon Denis Poisson

(1781-1840)France

"Life is good for only two things: discovering mathematics and teaching mathematics."

Poisson Process

- **Wikipedia:** A **Poisson process**, named after the French mathematician **Siméon-Denis Poisson** (1781 – 1840), is the **stochastic process** in which events (e.g. arrivals) occur continuously and **independently of one another**.
- **Formal Definition:** The Poisson Process is a **counting function** $\{N(\tau), \tau \geq 0\}$ where $N(\tau)$ is the number of events that have occurred up to time t , i.e. in the interval $[0, \tau]$.
- **Fact:** The number of events between time *a* and time *b* is given as $N(b) - N(a)$ and has a **Poisson distribution**.
- The Poisson process is a **continuous-time process**: Time is **continuous**
  - Its discrete-time counterpart is the *Bernoulli process*
    - Bernoulli process is a discrete-time stochastic process consisting of a sequence of independent random variables taking values over two symbols.

2

Examples of using Poisson Process

- The number of web page requests arriving at a server may be characterized by a Poisson process except for unusual circumstances such as coordinated denial of service attacks.
- The number of telephone calls arriving at a switchboard, or at an automatic phone-switching system, may be characterized by a Poisson process.
- The number of raindrops falling over a wide spatial area may be characterized by a spatial Poisson process.
- The arrival of "customers" is commonly modelled as a Poisson process in the study of simple queueing systems.
- The execution of trades on a stock exchange, as viewed on a tick by tick basis, is a Poisson process.

3

Applications of Poisson$_{\text{Review}}$

- **Context:** number of events occurring in a fixed period of time
  - Events occur with a known average rate and are independent
- Possion distribution is characterized by the **average rate** λ
  - The average number of arrival in the fixed time period.

- **Examples**
  - The number of cars passing a fixed point in a 5 minute interval. Average rate: λ = 3 cars/5 minutes
  - The number of calls received by a switchboard during a given period of time. Average rate: λ =3 call/minutes
  - The number of message coming to a router per second
  - The number of travelers arriving to the airport for flight registration

Poisson Distribution Review

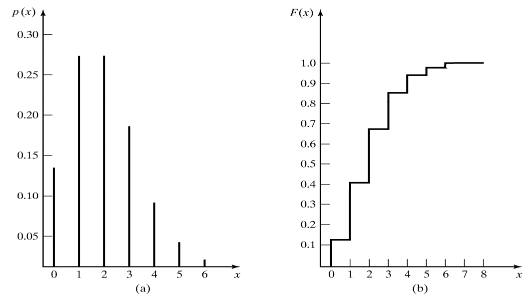
- The Poisson distribution with the average rate parameter λ

PMF: $p(k) = P(X = k) = \begin{cases} \frac{\lambda^k}{k!}\exp(-\lambda) & \text{for } k = 0,1,2, \ldots \\ 0, & \text{otherwise} \end{cases}$

CDF: $F(k) = p(X \le k) = \sum_{i=0}^{k} \frac{\lambda^i}{i!} \cdot \exp(-\lambda)$

Expected value: $E[X] = \lambda$

Variance: $V[X] = \lambda$



Poisson R.V. Review

- Ex. A production line with .4 percent of its items are defective , n=500 items are taken for a quality control. What is the probability that 0, 1, 3 items of them are defective
- That is X=b(500,0.004) aprox. To Poisson

$$P(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda}$$
$$\lambda = 500 * .004 = 2$$
$$P(x = 0) = e^{-2}$$
$$P(x = 1) = 2e^{-2}$$
$$P(x = 3) = \frac{4}{3} e^{-2}$$

Example: Poisson Review

- The number of cars that enter the parking follows a Poisson distribution with a mean rate equal to λ = 20 **cars/hour**
  - The probability of having **exactly** **15 cars** entering the parking in **one hour**:

$$p(15) = P(X = 15) = \frac{20^{15}}{15!} \cdot \exp(-20) = 0.051649$$

Example: Poisson Review

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).
  - The probability of three beeps in the next hour:

$$P(X = 3) = \frac{(\lambda)^k}{k!} e^{-\lambda} = \frac{(2)^3}{3!} e^{-2} = 0.18$$

$$Or$$

$$P(X = 3) = F(3) - F(2) = 857 - .677 = 0.18$$

  - The probability of two or more beeps in a 1-hour period:

$$P(X >= 2) = 1 - p(0) - p(1) = 1 - F(1) = 0.594$$

8

Stationary Poisson Process

- *Also known as* ***homogeneous*** **Poisson process** is characterized by a **CONSTANT rate parameter λ**, also known as **intensity**, such that the number of events in time interval $(t, t+\tau]$ follows a Poisson distribution with associated parameter $\lambda \cdot \tau$.
- Formally, A counting process {*N* (*t*), *t* ≤ 0} is a (Stationary) Poisson process with mean rate λ if:

$$\text{for } t \geq 0 \text{ and } n = 0, 1, 2, \ldots$$

$$\text{PMF:} \quad p\left[N(t+\tau) - N(t) = n\right] = p\left[N(\tau) = n\right] = \frac{(\lambda \cdot \tau)^n}{n!} \exp(-\lambda \cdot \tau)$$

- $N(t+\tau) - N(t)$ describes the number of events in time interval $(t, t+\tau]$
- The mean and the variance are equal

$$E\left[N(\tau)\right] = V\left[N(\tau)\right] = \lambda \cdot \tau$$

9

Stationary Poisson Process

- **Properties of Poisson process**
  - Arrivals occur one at a time (not simultaneous)
  - $\{N(\tau), \tau \geq 0\}$ has **stationary increments**, which means

$$N(t) - N(s) = N(t-s)$$

The number of arrivals in time *s* to *t* is also Poisson-distributed with mean

  - $\lambda \cdot (t-s)$ has **independent increments**

$\{N(\tau), \tau \geq 0\}$

10

Generating a Poisson Process

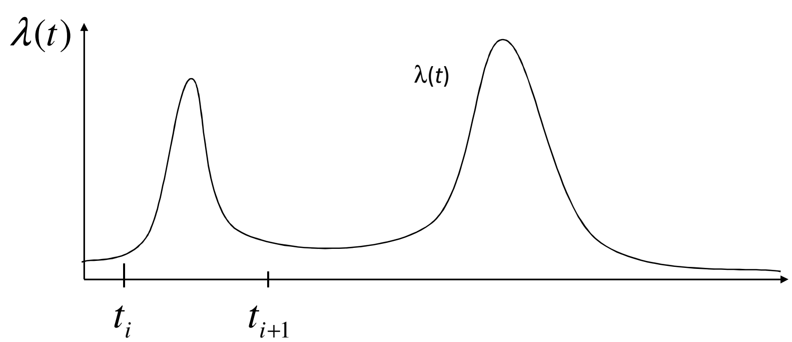
- Stationary with rate $\lambda>0$
- Time between events $A_i=t_i-t_{i-1}$ are IID exponential
- Algorithm

$$1.\ \text{Generate}\ U \sim U(0,1)$$
$$2.\ \text{Return}\ t_i = t_{i-1} - \left(1/\lambda\right)\ln U$$

11

Nonstationary Case

- Can we simply generalize?



12

Thinning Algorithm

1. Set $t=t_{i-1}$
2. Generate U_1, U_2 I.I.D. U(0,1)
3. Replace t by

$$t-\left(\frac{1}{\lambda^*}\right)\ln U_1, \text{ where } \lambda^* = \max_t\{\lambda(t)\}$$

4. If $U_2 \le \lambda(t)/\lambda^*$ return $t_i = t$. Otherwise, go back to step 2.

13

(Homogenous) Poisson Process

- *The* ***homogeneous*** **Poisson process** is characterized by a **CONSTANT rate parameter λ**, also known as **intensity**, such that the number of events in time interval $(t, t+\tau]$ follows a Poisson distribution with associated parameter $\lambda \cdot \tau$.
- Formally, A counting process $\{N(\tau), \tau \ge 0\}$ is a (homogenous) Poisson process with mean rate λ if:

for $t \ge 0$ and $n = 0, 1, 2, \ldots$

PMF: $p\left[N(t+\tau)-N(t)=n\right]=p\left[N(\tau)=n\right]=\frac{(\lambda \cdot \tau)^n}{n!}\exp(-\lambda \cdot \tau)$

- $N(t+\tau)-N(t)$ describes the number of events in time interval $(t, t+\tau]$
- The mean and the variance are equal $E\left[N(\tau)\right]=V\left[N(\tau)\right]=\lambda \cdot \tau$

14

(Homogenous) Poisson Process

- **Properties of Poisson process**
  - Arrivals occur one at a time (not simultaneous)
  - $\{N(\tau), \tau \ge 0\}$ has **stationary increments**, which means

$$N(t) - N(s) = N(t - s)$$

The number of arrivals in time s to t is also Poisson-distributed with mean $\lambda \cdot (t - s)$

  - $\{N(\tau), \tau \ge 0\}$ has **independent increments**

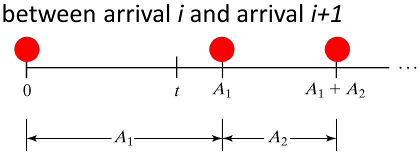
15

CDF of Exponential distribution

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & , x \ge 0, \\ 0 & , x < 0. \end{cases}$$

Inter-Arrival Times of a Poisson Process

- **Inter-arrival time:** time between two consecutive arrivals
  - The inter-arrival times of a Poisson process are random. ***What is its distribution?***
- Consider the **inter-arrival times** of a Poisson process (A_1, A_2, …), where A_i is the elapsed time between arrival *i* and arrival *i+1*



- The first arrival occurs after time t **MEANS** that there are no arrivals in the interval [0,t], As a consequence:

$$p(A_1 > t) = p(N(t) = 0) = \exp(-\lambda \cdot t)$$

$$p(A_1 \le t) = 1 - p(A_1 > t) = 1 - \exp(-\lambda \cdot t)$$

B1

The Inter-arrival times of a Poisson process are exponentially distributed and independent with mean $1/\lambda$

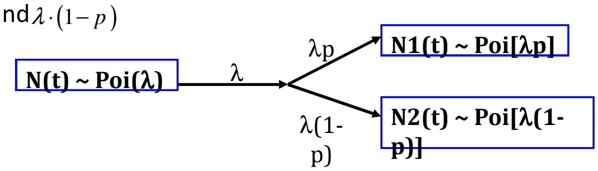
16

Slide 16

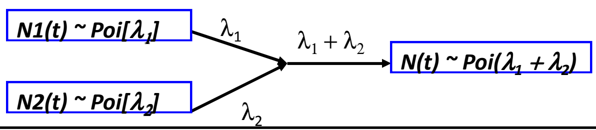
B1 Poi is not an abbreviation of Poisson that I have ever seen
Brian, 07/01/2005

Splitting and Pooling

- **Splitting**
  - A Poisson process can be split into two Poisson processes: The first with a probability *p* and the second with probability *1-p.*
  - $N(t) = N_1(t) + N_2(t)$ where $N_1(t)$ and $N_2(t)$ are both Poisson processes with rates $\lambda \cdot p$ and $\lambda \cdot (1-p)$



- **Pooling**
  - The summation of two Poisson processes is a Poisson process
  - $N_1(t) + N_2(t) = N(t)$ where $N(t)$ is a Poisson processes with rates $\lambda_1 + \lambda_2$



Non Homogenous (Non-stationary) Poisson Process (NSPP)

- *The* ***non homogeneous*** **Poisson process** is characterized by a **VARIABLE rate parameter λ(t)**, the arrival rate at time *t*. In general, the rate parameter may change over time.

- The stationary increments, property is not satisfied

$$\exists s,t: \; N(t) - N(s) \neq N(t-s)$$

- The **expected number of events** (e.g. arrival) between time *s* and time *t* is

$$\lambda_{s,t} = \int_s^t \lambda(u) \cdot du$$

18