

Satisfiability

- A sentence is **satisfiable** if it is true under some interpretation (i.e. it has a model), otherwise the sentence is **unsatisfiable**.
- A sentence is **valid** if and only if its negation is unsatisfiable.
- Therefore, algorithms for either validity or satisfiability checking are useful for logical inference.
- If there are n propositional symbols in a sentence, then we must check 2^n rows for validity
- **Satisfiability is NP-complete**, i.e. there is no polynomial-time algorithm to solve.
- Yet, many problems can be solved very quickly.

Pros and cons of propositional logic

- ✓ Propositional logic is declarative:
pieces of syntax correspond to facts
- ✓ Propositional logic is compositional:
meaning of $A \wedge B$ is derived from meaning of A and B
- ✓ Meaning in propositional logic is context-independent
 - (unlike natural language, where meaning depends on context)
- Propositional logic has very limited expressive power
 - (unlike natural language)

First-order logic

- First-order logic (FOL) models the world in terms of
 - Objects, which are things with individual identities
 - Properties of objects that distinguish them from other objects
 - Relations that hold among sets of objects
 - Functions, which are a subset of relations where there is only one “value” for any given “input”

Ex: Objects: Students, lectures, companies, cars ...

- Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
- Properties: blue, oval, even, large, ...
- Functions: father-of, best-friend, second-half, one-more-than ...

A common mistake to avoid

- Typically, \Rightarrow is the main connective with \forall
- Common mistake: using \wedge as the main connective with \forall :

- Ex:

$\forall x \text{ At}(x, \text{CU}) \wedge \text{Smart}(x)$

means “Everyone is at CU and everyone is smart”

Yet to say Everyone at CU is smart

$\forall x \text{ At}(x, \text{CU}) \Rightarrow \text{Smart}(x)$

Another common mistake to avoid

- Typically, \wedge is the main connective with \exists
- Common mistake: using \Rightarrow as the main connective with \exists :

$$\exists x \text{At}(x, \text{CU}) \Rightarrow \text{Smart}(x)$$

is true if there is anyone who is smart not at CU.

Yet to say: there exists someone in CU that is smart

$$\exists x \text{At}(x, \text{CU}) \wedge \text{Smart}(x)$$

Examples of FOPC

- Brothers are siblings

$$\forall x, \forall y \text{ Brother}(x, y) \Rightarrow \text{Sibling}(x, y)$$

- One's mother is one's female parent

$$\forall m, \forall c \text{ Mother}(c) = m \Leftrightarrow (\text{Female}(m) \wedge \text{Parent}(m, c))$$

- "Sibling" is symmetric

$$\forall x, \forall y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x)$$

Translating English to FOL

- Every gardener likes the sun.

$(\forall x) \text{gardener}(x) \Rightarrow \text{likes}(x, \text{Sun})$

Translating English to FOL

- Every gardener likes the sun.

$(\forall x) \text{gardener}(x) \Rightarrow \text{likes}(x, \text{Sun})$

- You can fool some of the people all of the time.

$(\exists x) \text{person}(x) \wedge ((\forall t) \text{time}(t)) \Rightarrow \text{can-fool}(x, t)$

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- All purple mushrooms are poisonous.

$(\forall x) (\text{mushroom}(x) \wedge \text{purple}(x)) \Rightarrow \text{poisonous}(x)$

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- All purple mushrooms are poisonous.

$(\forall x) (\text{mushroom}(x) \wedge \text{purple}(x)) \Rightarrow \text{poisonous}(x)$

- No purple mushroom is poisonous.

$\sim(\exists x) \text{purple}(x) \wedge \text{mushroom}(x) \wedge \text{poisonous}(x)$

or, equivalently,

$(\forall x) (\text{mushroom}(x) \wedge \text{purple}(x)) \Rightarrow \sim\text{poisonous}(x)$

Translating English to FOL

- There are exactly two purple mushrooms.

$(\exists x) (\exists y) \text{mushroom}(x) \wedge \text{purple}(x) \wedge \text{mushroom}(y) \wedge$
 $\text{purple}(y) \wedge \sim(x=y) \wedge (\forall z) (\text{mushroom}(z) \wedge \text{purple}(z))$
 $\Rightarrow ((x=z) \vee (y=z))$

Inference in FOL

- $KB \vdash_i \alpha$ = sentence α can be derived from KB by procedure i
i.e. deriving sentences from other sentences
- **Soundness**: i is sound if whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
i.e. derivations produce only entailed sentences (*no wrong inferences, but maybe not all inferences*)
- **Completeness**: i is complete if whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$
i.e. derivations can produce all entailed sentences (*all inferences can be made, but maybe some wrong extra ones as well*)

Validity and satisfiability

- A sentence is **valid** if it is true in **all models**,
- e.g., **True**, $A \vee \neg A$, $A \Rightarrow A$, $(A \wedge (A \Rightarrow B)) \Rightarrow B$

Validity is connected to inference via the following:

$KB \vdash \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid

A sentence is **satisfiable** if it is true in **some model**

e.g., $A \vee B$, C

A sentence is **unsatisfiable** if it is true in **no models**

e.g., $A \wedge \neg A$

Satisfiability is connected to inference via the following:

$KB \not\vdash \alpha$ if and only if $(KB \wedge \neg \alpha)$ is unsatisfiable

(there is no model for which $KB = \text{true}$ and α is false)

Proof Methods in FOL

Major Families:

- GMP
- Reduction
- Resolution
- Forward chaining
- Backward chaining

Some Other inference tools:

Entailment/ Unification/

Proof Methods in FOL

- GMP: Using the generalized form of Modus Ponense
- Reduction: Reduce all FOL sentences to propositional Calculus then use inference in propositional calculus
- Resolution – Refutation
 - Negate goal
 - Convert all pieces of knowledge into clausal form (disjunction of literals)
 - See if contradiction indicated by null clause \square can be derived
- Forward chaining
 - Given $P, P \rightarrow Q$, to infer Q
 - P , match *L.H.S* of
 - Assert Q from *R.H.S*
- Backward chaining
 - Q , Match *R.H.S* of $P \rightarrow Q$
 - assert P
 - Check if P exists

Universal instantiation (UI)

- Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v \alpha}{\text{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g

- E.g., $\forall x \text{ King}(x) \wedge \text{Greedy}(x) \Rightarrow \text{Evil}(x)$ yields:
 $\text{King}(\text{John}) \wedge \text{Greedy}(\text{John}) \Rightarrow \text{Evil}(\text{John})$
 $\text{King}(\text{Richard}) \wedge \text{Greedy}(\text{Richard}) \Rightarrow \text{Evil}(\text{Richard})$
 $\text{King}(\text{Father}(\text{John})) \wedge \text{Greedy}(\text{Father}(\text{John})) \Rightarrow \text{Evil}(\text{Father}(\text{John}))$

Existential instantiation (EI)

- For any sentence α , variable v , and constant symbol k that does **not** appear elsewhere in the knowledge base:

$$\frac{\exists v \alpha}{\text{Subst}(\{v/k\}, \alpha)}$$

- E.g., $\exists x \text{ Crown}(x) \wedge \text{OnHead}(x, \text{John})$ yields:

$$\text{Crown}(C_7) \wedge \text{OnHead}(C_7, \text{John})$$

provided C_7 is a new constant symbol, called a **Skolem constant**

Unification

- $\forall x \text{ King}(x) \wedge \text{Greedy}(x) \Rightarrow \text{Evil}(x)$
- We can get the inference immediately if we can find a substitution θ such that $\text{King}(x)$ and $\text{Greedy}(x)$ match $\text{King}(\text{John})$ and $\text{Greedy}(y)$

$\theta = \{x/\text{John}, y/\text{John}\}$ works

- $\text{Unify}(a, \beta) = \theta$ if $a\theta = \beta\theta$

p	q	θ
Knows(John,x)	Knows(John,Jane)	
Knows(John,x)	Knows(y,OJ)	
Knows(John,x)	Knows(y,Mother(y))	
Knows(John,x)	Knows(x,OJ)	

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p	q	θ
Knows(John,x)	Knows(John,Jane)	$\{x/Jane\}$
Knows(John,x)	Knows(y,OJ)	$\{x/OJ, y/John\}$
Knows(John,x)	Knows(y,Mother(y))	$\{y/John, x/Mother(John)\}$
Knows(John,x)	Knows(x,OJ)	$\{fail\}$

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Unification

- To unify $Knows(John, x)$ and $Knows(y, z)$,
 $\theta = \{y/John, x/z\}$ or $\theta = \{y/John, x/John, z/John\}$ or others...
- There are many possible unifiers for some atomic sentences. The first unifier is more general than the second.
- The UNIFY algorithm returns the most general unifier (MGU) that is unique up to renaming of variables. MGU makes the least commitment to variable values.

The Unification Algorithm

- In order to match sentences in the KB, we need a routine.
 - UNIFY(p,q) takes two atomic sentences and returns a substitution that makes them equivalent.
- UNIFY(p,q) = θ where SUBST(θ ,p)=SUBST(θ ,q) θ is called a unifier.

```
function UNIFY(x, y,  $\theta$ ) returns a substitution to make x and y identical
  inputs: x, a variable, constant, list, or compound
         y, a variable, constant, list, or compound
          $\theta$ , the substitution built up so far

  if  $\theta = \text{failure}$  then return failure
  else if  $x = y$  then return  $\theta$ 
  else if VARIABLE?(x) then return UNIFY-VAR(x, y,  $\theta$ )
  else if VARIABLE?(y) then return UNIFY-VAR(y, x,  $\theta$ )
  else if COMPOUND?(x) and COMPOUND?(y) then
    return UNIFY(ARGs[x], ARGs[y], UNIFY(OP[x], OP[y],  $\theta$ ))
  else if LIST?(x) and LIST?(y) then
    return UNIFY(REST[x], REST[y], UNIFY(FIRST[x], FIRST[y],  $\theta$ ))
  else return failure
```

The Unification Algorithm

```
function UNIFY-VAR(var, x,  $\theta$ ) returns a substitution
  inputs: var, a variable
         x, any expression
          $\theta$ , the substitution built up so far

  if  $\{var/val\} \in \theta$  then return UNIFY(val, x,  $\theta$ )
  else if  $\{x/val\} \in \theta$  then return UNIFY(var, val,  $\theta$ )
  else if OCCUR-CHECK?(var, x) then return failure
  else return add  $\{var/x\}$  to  $\theta$ 
```