Quiz on Sun. May 14 on Generating Random Numbers / Variates

Convolution

- Assume that $X = Y_1 + Y_2 + ... + Y_m$ (where the ***Y***s are IID with CDF), called m-fold convolution of the distributions of Y's
- Algorithm
  1. Generate $Y_1, Y_2, ..., Y_m$ IID each with CDF
  2. Return $X = Y_1 + Y_2 + ... + Y_m$

Erlang-*m* Distribution

The Erlang-*m* random variable with mean β, then

$$X = Y_1 + Y_2 + \ldots + Y_m \ ,$$

where the Y_i's are IID exponential random variables, each with mean β/m. If we use the inverse-transform method to generate the exponential Y_i's, then

$$X = \sum_{i=1}^{m} Y_i = \sum \frac{-\beta}{m} \ln U_i = \frac{-\beta}{m} \ln\left(\prod_{i=1}^{m} U_i\right).$$

Erlang-*m* Distribution

Hence we generate random variate *x* as follows:

1. Generate u_1, u_2, …, u_m as IID $U(0,1)$.
2. Return $x = \frac{-\beta}{m} \ln\left(\prod_{i=1}^{m} u_i\right)$.

Acceptance-Rejection Method

By Von Neumann (1951), used when the direct approaches fail or inefficient.

A closed form for F(X) does not exist, so what we'll do is to add another distribution . For which we know "how to calculate the CDF and its inverse".

We pick a function t(x) that is larger than f(x) for all x. Technically we say that t(x) ***majorizes*** f(x).

Acceptance-Rejection Method

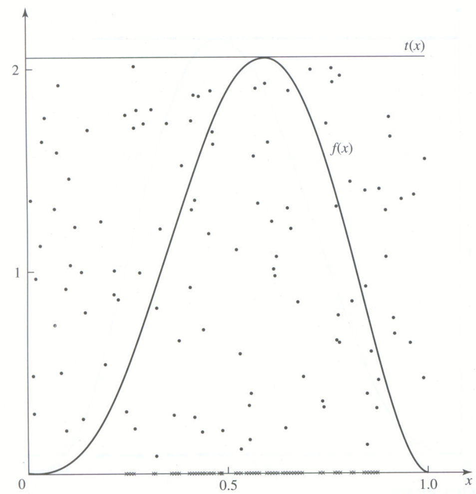
- Specify a function that majorizes the density

$$t(x) \geq f(x), \forall x$$

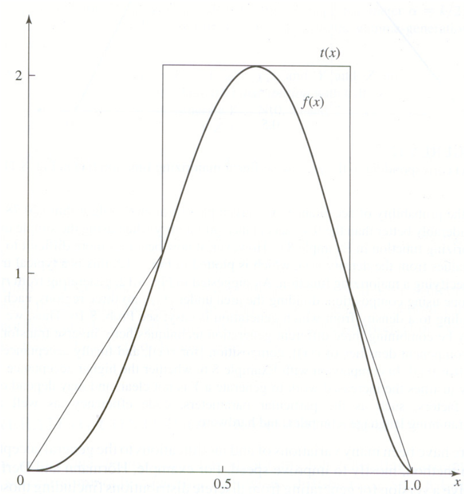
- ***t(x) is clearly not a density fun.***
- New density function $r(x) = t(x) \Big/ \int_{-\infty}^{\infty} t(x)dx$
- Algorithm to generate a r.v. with density ***f(x)***

1. Generate Y with density r
2. Generate U independint of Y
3. If $U \leq f(Y)/t(Y)$, return $X = Y$.
   Otherwise go back to Step 1.

Example:



Example: More Efficient



Beta Distribution

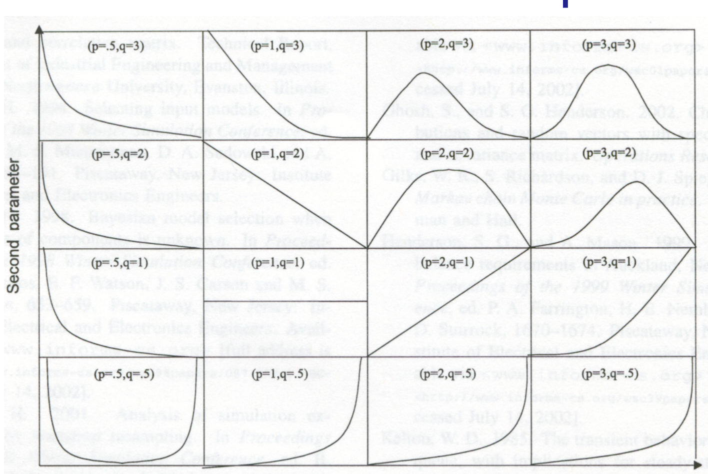
- Density

$$f(x) = \begin{cases} \dfrac{x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$B(\alpha_1, \alpha_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1}\,dt$$

- No closed form CDF. No closed form inverse
- Must use numerical methods for inverse-transform method

Beta Distribution Shapes



Beta Density shapes

Beta(4,3) Example

- Density $f(x) = \begin{cases} 60x^3(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$
- The distribution ***F(x)*** is a 6th degree polynomial, The inverse approach is not a good one.
- Try to find ***t(x)***

  put ***df/dx =0 to*** find the max of the f, which will be a ***x =0.6*** , and ***f(0.6)=2.0736***
- Put

$$t(x) = \begin{cases} 2.0736 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Then $r(x) = \dfrac{t(x)}{\int_{-\infty}^{\infty} t(x)dx} = \dfrac{2.0736}{\int_0^1 2.0736dx} = 1$

Beta(4,3) Example

So ***r(x)*** is just ***U(0,1)***

The algorithm:

1. ***Generate Y having density of r(x) ; i.e. U(0,1)***
2. ***Generate U ~ U(0,1), independent of Y***
3. ***If*** $U \le \dfrac{f(Y)}{t(Y)}$ ***, return X=Y; Otherwise go to step 1***

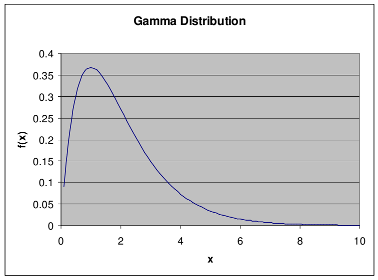
Acceptance – Rejection Technique for Gamma Dist.

A gamma(α,β) density

$$f(x) = \begin{cases} \dfrac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)} & 0 < x \\ 0 & \text{otherwise} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-1} dy$$

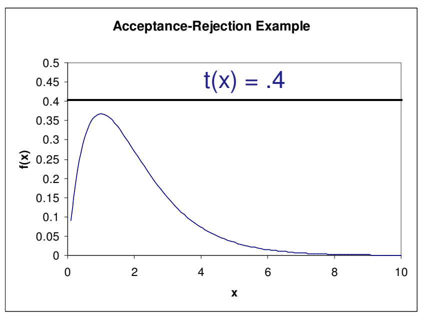


A closed form for F(X) does not exist,

Acceptance – Rejection Technique for Gamma Dist.

A gamma(2,1)

We pick a function t(x) that is larger than f(x) for all x. t(x)=0.4



Acceptance-Rejection Gamma(2,1)

In we selected t(x) = .4 0 <= x <= 10

Now t(x) is not a density function, why?

since its integral from 0 to 10 doesn't add up to 1.

So let us define c: And r(x) = t(x)/c:

$$c = \int_0^\infty t(x)\,dx$$
$$= \int_0^{10} .4\,dx$$
$$= \Big|_0^{10} .4x$$
$$= 4$$

$$r(x) = .4/4$$
$$= .1$$

$$R(x) = \int_0^x .1dx'$$
$$= .1x'$$

i.e. ***r(x)*** is a density fn.

Acceptance-Rejection

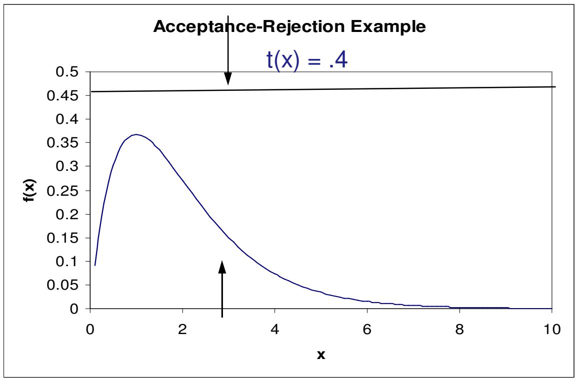
And in this simple case, we can easily determine the inverse transformation for R: X = 10Y.

So, let's say we pick a random number Y = .3.
This translates into an X of 3

Acceptance – Rejection

The inverse transformation for R: X = 10Y.

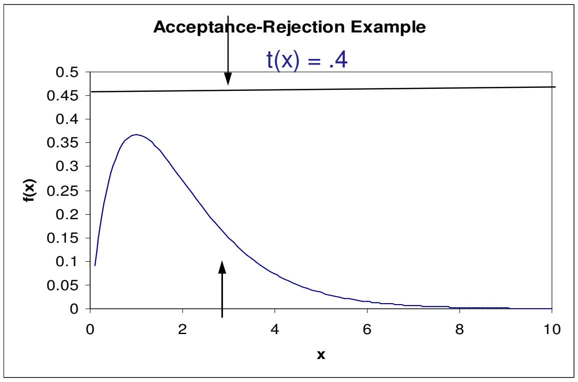
For a random number Y = 0.3,
This translates into an X of 3



If we threw darts that could land only on the line x = 3, then the probability that a dart hitting inside the distribution would be f(X=3)/t(X=3).

Acceptance – Rejection

f(X=.3)/t(X=.3)
= .15/.4
= .375



Generate U~U(0,1).

If U is less than .375, we will accept X= 3 as coming from a gamma(2,1) distribution.

Otherwise, we will start the process over by selecting a new R and new U.

Derived Distributions

- Several distributions are derived from the gamma and normal
- Can take advantage of knowing how to generate those two distributions

Outline of the course

- Introduction to Simulation.
- Hand Simulation.
- Review of basic Probability Theory.
- Random Number Generation
- Generation of Random Varieties.
- Analysis of Output.
- Elementary Queuing Models

Modeling and Simulation

Model

- It is a simplification of the reality
- A (usually miniature) representation of an actual system; an example for imitation or emulation
- A description of observed behavior, simplified by ignoring certain details.
- Models allow complex systems to be understood and their behavior predicted within the scope of the model
- A model can be **Analytical** (Queuing Theory) or by Simulation.

Objectives

- To encourage "system" thinking
- Provide background to systems modeling concepts
- Opportunity for a practical appreciation for discrete event simulation
- Combine theory and practice

Applications of Queuing Theory

- Telecommunications
- Computer Networks
- Predicting computer performance
- Health services (eg. control of hospital bed assignments)
- Airport traffic, airline ticket sales
- Layout of manufacturing systems.

Discrete-Event Simulation

- Discrete Event System
  - a system whose ***state*** changes at discrete points in time due to the occurrence of asynchronous ***events***
- Example: M/M/1 queuing system
  - State
    - number of customers in system
  - Events
    - customer arrival
    - customer departure

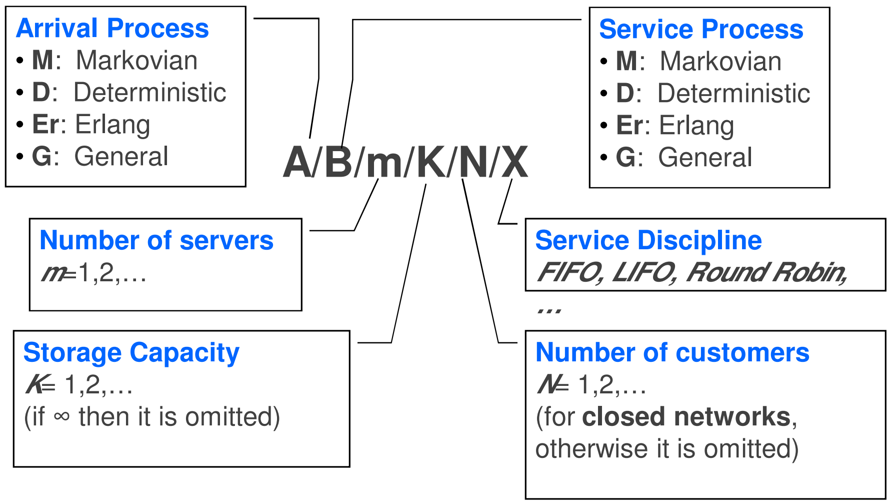
Queuing System

- A system Characterized by three components
  - Arrival Process The distribution that determines how the tasks arrives in the system).
  - Service Process The distribution that determines the task processing time
  - Number of Servers Total number of servers available to process the tasks

M/M/1 System

- Nomenclature: M stands for "Memoryless" (a property of the exponential distribution)
  - **M**/M/1 stands for Poisson arrival process (which is memoryless)
  - M/**M**/1 stands for exponentially distributed transmission times
  - M/M/1 stands for one server
- "M/M/1" is a special case of more general (Kendall) notation: X/Y/m/k
- Example of packet :
  - Arrival process is Poisson with rate l
  - Processing times are exponentially distributed with mean 1/m
  - One server
  - Independent interarrival times and processing times

Kendall Notation



Kendall Notation

- M/M/1 Queue Poisson arrivals (***exponential inter-arrival***), and exponential service, 1 server, infinite capacity and population, FCFS (FIFO). The most basic and important queuing model -Was the subject of our course
- M/M/m Queue ***Same, but m servers***
- M/M/m/k Queue system Same as M/M/m, but there is buffer space for at most k packets. Packets arriving at a full buffer are dropped, k is omitted when k = ∞
- M/D/1 Queue Poisson arrivals and CONSTANT service times, 1 server, infinite capacity and population, FIFO.
- G/G/3/20/1500/SPF General arrival and service distributions, 3 servers, 17 queues (20-3), 1500 total jobs, Shortest Packet First