Chapter 8
Generating Random Variate

- If a Pseudo random number Y is used to generate another variable X, then X is called random variate.
- Note that the sequence X may or may not be uniform.
- Generally, all random variates require the use of a U(0,1) distribution.

Random Variate

- The exact technique used to generate an observation varies with the type of distribution used. However, in general we're looking for techniques that are:
  - Exact
  - Efficient in terms of time, storage, and setup
  - Low in complexity
  - Robust.

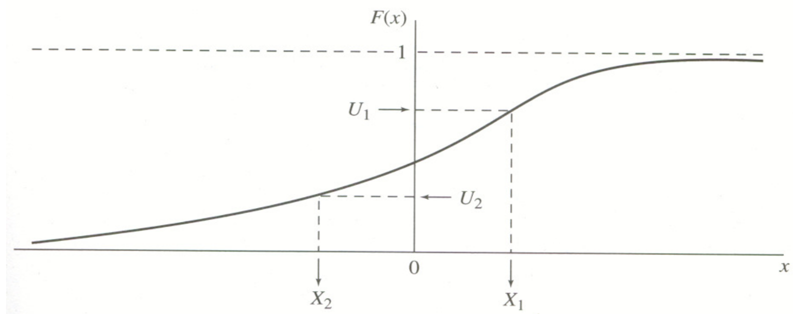
Generating Random Variates

- Say we have fitted an exponential distribution to inter arrival times of customers
- Every time we anticipate a new customer arrival (place an arrival event on the events list), we need to generate a realization of the arrival times
- Already know how to generate unit uniform
- Can we use this to generate exponential? (And other distributions)

Two Types of Approaches

- Direct
  - Obtain an analytical expression
  - Inverse transform
    - Requires inverse of the distribution function
  - Composition & Convolution
    - For special forms of distribution functions
- Indirect
  - Acceptance-rejection

Inverse-Transform Method



- Place r_i on the y-axis and find the corresponding x_i

$$r = F(x) \Rightarrow x = F^{-1}(r)$$

Formulation

- Algorithm

  1. Generate $U \sim U(0,1)$

  2. Return $X = F^{-1}(U)$

- Proof Since $F(x)$ is a monotone function, then

$$P(X \le x) = P(F^{-1}(U) \le x)$$
$$= P(U \le F(x))$$
$$= F(x)$$

Example

- Let

$$f(x)=\begin{cases}4x^3 & 0<x\le 1\\ 0 & O.W.\end{cases}$$

$$F(x)=\int_0^x 4x^3dx = x^4$$

- i.e.

$$r = x^4$$

$$x_i = F^{-1}(r_i)=\sqrt[4]{r_i} \qquad 0<r_i\le 1$$

- That is use a pseudo random generator to find r_i then find x_i for the new random variate

Ex. Uniform random variate

- Let

$$f(x)=\begin{cases}\dfrac{1}{b-a} & a<x\le b\\ 0 & O.W.\end{cases}$$

- i.e.

$$F(x)=\frac{x-a}{b-a}=r \qquad 0<r\le 1$$

$$x = a + r(b-a)$$

- Use a pseudo random generator to find r_i then find x_i for the new random variate uniformly distributed on (a,b].

Exponential R.V.

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & \lambda>0, x\geq 0\\ 0 & O.W.\end{cases}$$

$$F(x)=\begin{cases}1-e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$$

$$r=1-e^{-\lambda x} \qquad 0<r\leq 1$$

$$e^{-\lambda x}=1-r$$

$$x=\frac{-1}{\lambda}\ln(1-r)$$

Exponential R.V.

- Algorithm:

$$1.\ \text{Generate } U \sim U(0,1)$$

$$2.\ \text{Return } X=\frac{-1}{\lambda}\ln(1-U)$$

Exponential R.V.

- Algorithm:

$$1.\ \text{Generate}\ U \sim U(0,1)$$

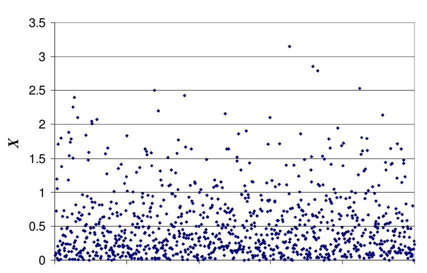
$$2.\ \text{Return}\ X_U = \frac{-1}{\lambda}\ln(1-U)$$

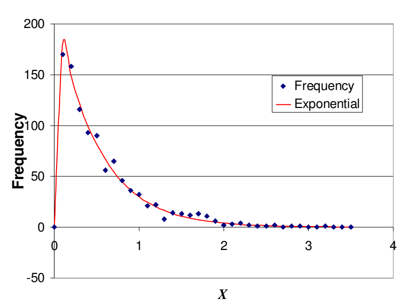
- Remark: Since $U \sim U(0,1)$ then also $1-U \sim U(0,1)$

And hence for sake of computation we may replace ***1-U*** by ***U*** in the above algorithm.

Exponential Distribution

1000 Exponential Random Numbers (λ = 2)





Exponential R.V.

- Example:

- For a sample of 5 numbers and $\lambda = 3.8$
- Use a R.N.G. to find r_i's

r_i	0.135	0.639	0.424	0.01	0.843
ln r_i	-2.002	-0.448	-0.858	-4.61	-171
x_i	0.526	0.118	0.226	1.213	0.045

Exponential distribution

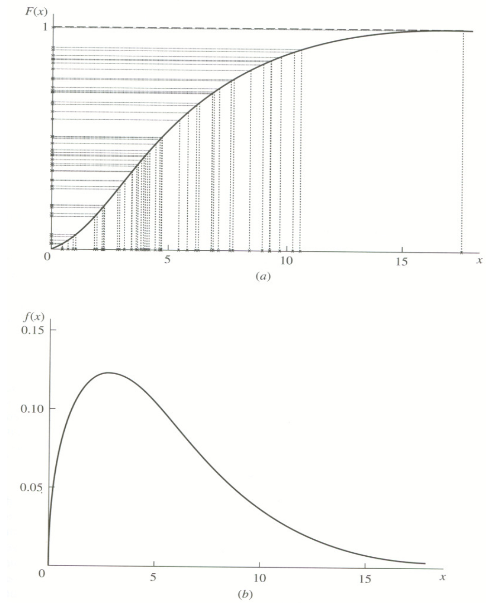
Needed for many applications

- Probability of waiting time on phone calls
- Radio active decay materials
- Time arrival of customers to some server
- Time of failure of machines/ components

Example: Weibull

- The CDF of a Weibull is

$$F(X) = 1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}$$



Inverse Transformation-Weibull

$$r = 1 - e^{-\left(\frac{X}{\alpha}\right)^{\beta}}$$

$$(1 - r) = e^{-\left(\frac{X}{\alpha}\right)^{\beta}}$$

$$\ln(1 - r) = -\left(\frac{X}{\alpha}\right)^{\beta}$$

$$\left[-\ln(1 - r)\right]^{\frac{1}{\beta}} = \frac{X}{\alpha}$$

$$\alpha\left[-\ln(1 - r)\right]^{\frac{1}{\beta}} = X$$

- Exr. Write the algorithm for generating a Weibull random variate

Advantages of Inverse Transform Method

- Straightforward method, easy to use
- Variance reduction techniques has an advantage if inverse transform method is used

Disadvantages of the inverse method

- Must evaluate the inverse of the distribution function
  - May not exist in closed form
  - Could still use numerical methods
- May not be the fastest way

Bernoulli

- Mass function

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

- Algorithm

1. Generate $U \sim U(0,1)$

2. If $U \leq p$ return $X = 1$. Otherwise return $X = 0$

Binomial

- Mass function

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$$

- Use the fact that if ***X***~B(***n***,***p***) then

$$X = Y_1 + Y_2 + \ldots, Y_n$$

$$Y_i \sim \text{Bernoulli}(p)$$

Binomial

If X is a binomial random variable with parameters n and p. Note the sum of n IID Bernoulli(p) random variables is a B(n, p) random variable.

To generate a binomial random variate,

1. Generate $y_1, y_2, \ldots, y_n$ as IID Bernoulli(p) random variates.
2. Set $x = y_1 + y_2 + \ldots + y_n$.

Geometric Distribution

$$p(x) = p(1-p)^x, \text{ where x} = 0, 1, 2, \ldots\ldots$$

$$F(x) = \sum_{j=0}^{x} p(1-p)^x = 1-(1-p)^{x+1}$$

$$R = 1-(1-p)^{x+1}$$

$$\Leftrightarrow (1-p)^{x+1} = 1-R$$

$$\Leftrightarrow (x+1)\ln(1-P) = \ln(1-R)$$

$$\Leftrightarrow x = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F^{-1}(R) = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F(x-1) < R \le F(x)$$

$$X = \lfloor F^{-1}(R) \rfloor = \left\lfloor \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rfloor$$

Geometric Distribution

- Algorithm to generate random variate for

  $$p(x) = p(1-p)^x, \text{ where x} = 0, 1, 2, \ldots\ldots$$

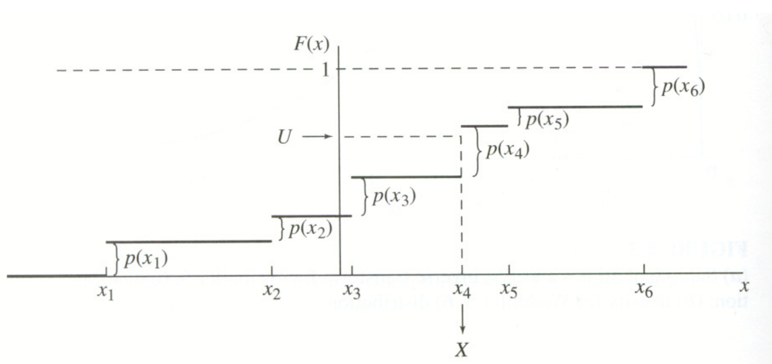
  – Generate `U~U(0,1)` uniform random number

  – Return $\left\lfloor \frac{\ln(1-U)}{\ln(1-p)} - 1 \right\rfloor$

- Remark

For simplicity you may return $X = \left\lfloor \frac{\ln U}{\ln(1-p)} \right\rfloor$

Discrete Distributions



Empirical Discrete Distribution

- Find the random variate for the following discrete distribution.

x	p(x)
0	0.50
1	0.30
2	0.20

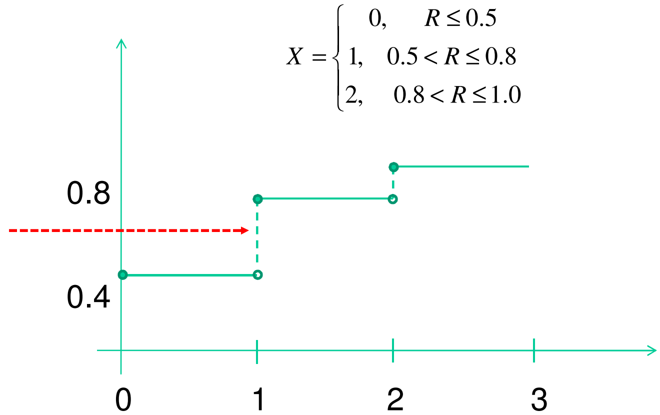
- At first find the F

x	p(x)	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

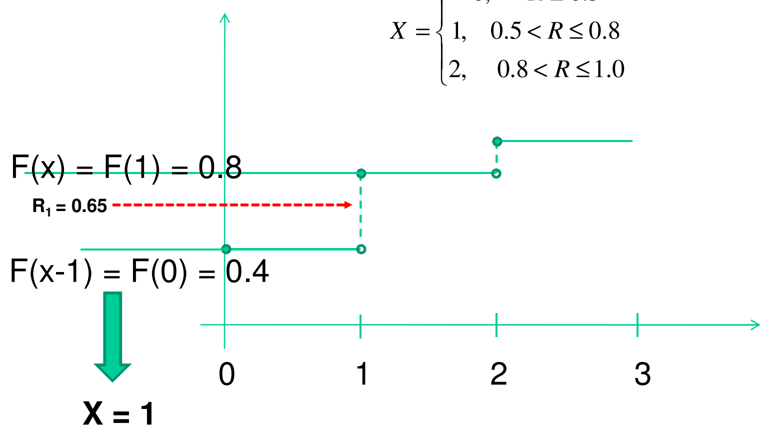
Empirical Discrete Distribution

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$



Empirical Discrete Distribution

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$



Example of Empirical Distribution

- There are four pumps(i-1 to 4) at a petrol station. And on average 1/3 of the customers used pump 1, 1/6 of them used pump 2, another 1/3 used pump 3 and the remaining 1/6 used pump 4. Outline a procedure for selection of pumps by various customers.

i	1	2	3	4
P(i)	1/3	1/6	1/3	1/6
F(i)	1/3	½	5/6	1

- The procedure:
- Generate r ~U(0,1)

  If $0 \le r < 1/3$ select pump 1

  If $1/3 \le r < 1/2$ select pump 2

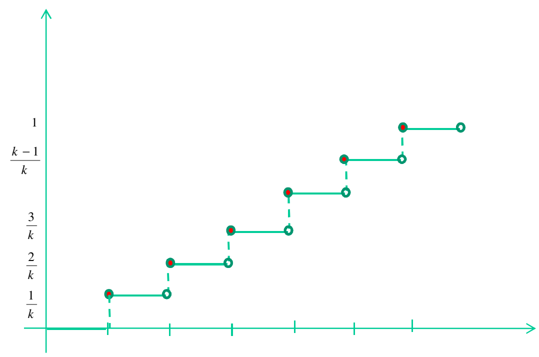
  If $1/2 \le r < 5/6$ select pump 3

  If $5/6 \le r < 1$ select pump 4

Discrete Uniform Distribution

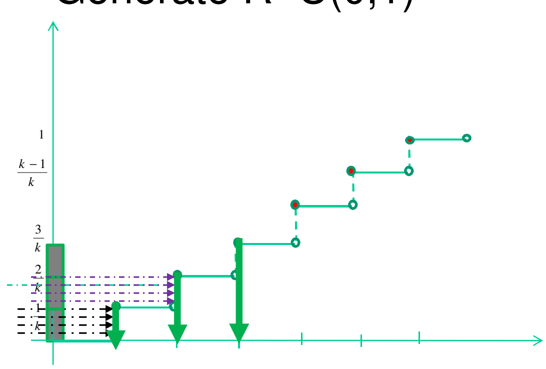
- Consider $p(x) = \frac{1}{k}, \; x = 1, 2, 3 \ldots k$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{k}, & 1 \le x < 2 \\ \frac{2}{k}, & 2 \le x < 3 \\ \vdots & \vdots \\ \frac{k-1}{k}, & k-1 \le x < k \\ 1, & k \le x \end{cases}$$



Discrete Uniform Distribution

- Generate R~U(0,1)



$$0 \le R \le \frac{1}{k}$$

$$\frac{1}{k} < R \le \frac{2}{k}$$

$$\frac{2}{k} < R \le \frac{3}{k}$$

$$\frac{3}{k} < R \le \frac{4}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

Discrete Uniform Distribution

$$0 \le R \le \frac{1}{k}$$

$$\frac{1}{k} < R \le \frac{2}{k}$$

$$\frac{2}{k} < R \le \frac{3}{k}$$

$$\frac{3}{k} < R \le \frac{4}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

$$\Rightarrow i-1 < Rk \le i$$

$$\Rightarrow i-1 < Rk \le i$$

$$\Rightarrow i < Rk+1 \text{ and } Rk \le i$$

$$\Rightarrow Rk \le i < Rk+1$$

$$\Rightarrow i = \lfloor RK \rfloor = \text{output } X$$

$$\therefore X = \lfloor RK \rfloor$$

Discrete Uniform Distribution

- Algorithm to generate random variate for p(x)=1/k where x = 1, 2, 3, … k
  - Generate R~U(0,1) uniform random number
  - Return $\lfloor RK \rfloor$