What is Monte Carlo Simulation ?

- **Monte Carlo methods** are a widely used class of **computational algorithms** for simulating the behavior of various physical and mathematical systems, and for other computations.
- **Monte Carlo algorithm** is often used to find solutions to mathematical numerical problems (which may have many variables) that cannot easily be solved, (e.g. integral calculus, or other numerical methods)

Monte Carlo Simulation

- A scheme employing random numbers which is used to solve certain stochastic or deterministic problems where the passage of time plays no substantive role.
- Common problem is the estimation of $\int_{\Omega} f(\mathbf{x})d\mathbf{x}$, where f is a function, **x** is a vector and Ω is domain of integration.
- Special case: Estimate $\int_a^b f(x)dx$ for scalar x and limits of integration a, b

Monte Carlo Simulation

Let X be a uniform random variable on the interval $[a, b]$ with density

$$p(x) = \frac{1}{b-a}, \quad a \le x \le b$$

and let $x_1, \ldots, x_n$ be a random sample from X. Then

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b \frac{f(x)}{p(x)} p(x)dx \\ &= (b-a)\int_a^b f(x)p(x)dx \\ &= (b-a)E[f(X)] \\ &\approx \frac{b-a}{n}\sum_{i=1}^n f(x_i)\,.\end{aligned}$$

Monte Carlo Simulation

Example: Estimate $\int_0^b \sin(x)dx$.

We approximate this by

$$\frac{b}{n}\sum_{i=1}^n \sin(x_i)\,,$$

where $x_1, \ldots, x_n$ are a sample from a uniform $[0, b]$ random variable.

Monte Carlo Simulation

Example: Estimate $\int_0^b \sin(x)dx$.

	n= 10	n= 100	n= 1000	n= 2000
b= 1 (answer = 2)	1.753	2.032	1.994	1.999
b= 2 (answer = 0)	-0.898	-0.013	0.137	0.079

There is considerable variability in the quality of solution; accuracy of numerical integration sensitive to integrand and domain of integration

Case Study
Cake's shop problem

An owner of a bakery shop would like to determine how many 10-inch birthday cakes he should produce each day in order to maximize his profit. His present method of determining the quantity to bake is based on his best guess.

Cake’s shop problem

- The production costs are \$2.00 per cake.
- And the profit for each cake is \$2.5.
- However, If over estimates the daily demand, some cakes will be left over at the end of the day. The policy is to sell all leftover cakes to a local store that specializes in day-old items. He is currently receiving \$1.50 per cake for the surplus cakes, thus incurring a loss of \$0.50 per cake.

Cake’s shop problem

- Case 1: The production quantity is less than or equal to demand

If $x \le d$, $z = 2.5x$

- Case 1: The production quantity is greater than the demand

If $x > d$ $z = 2.5d + (x-d)(-0.5)$

$Z = 3.00\,d - 0.5\,x$

Cake's shop problem

- Generalizeing:

p = selling price for each cake

c = cost of each unit

s = day-old price

- If x<= d, z = (p-c) x
- If x > d z = (p-c) d + (x-d) (s-c)

Z = (p-s) d – (s-c) x

Historical day demand for the birthday cakes

Daily deman	Frequency	Probability Distribution
0	1	0.05
1	2	0.1
2	1	0.05
3	2	0.1
4	3	0.15
5	6	0.3
6	3	0.15
7	1	0.05
8	1	0.05
Total	20	1

$$relative\_frequency = \frac{frequency\_of\_observation}{total\_number\_of\_observations}$$

Hand Simulation

- Take a sheet of paper and cut it into twenty equal pieces.
- Follow the historical daily demand frequency in the table,
- write the number zero on one piece.
- On two of the remaining pieces write the number one, which stands for the demand of one unit.
- Check the numbers you have written carefully, because this “deck” of twenty

Hand Simulation

- The first step is the selection of the production quantity, Assume (x=3).
- use the deck of twenty slips of paper to generate a demand by selecting one slip of paper at random.
- Suppose the first slip drawn has a 5 written on it.
- We shall then use a demand of 5 cakes for the first simulated day of bakery shop operation.
- i.e. underproduction of 2 cakes.

Hand Simulation

- Since x<d, we can computer our first day's profit using the expression
- 2.5x=25(3)=$7.5. i.e. Total profit of $7.5.
- Generate the demand for second day (reshuffle and draw a piece) suppose d=1
- Since x>d use the second case
- z=3(1) - 0.5(3) = $1.5
- So the total profit is 7.5 + 1.5 = 9

10-day simulation results for production quantity x=3

Day	Generated demand	Daily profit	Total profit
1	5	7.5	7.5
2	1	1.5	9
3	6	7.5	16.5
4	3	7.5	24
5	4	7.5	31.5
6	4	7.5	39
7	3	7.5	46.5
8	0	-1.5	45
9	5	7.5	52.5
10	6	7.5	60

Hand Simulation

- Now we perform the same ten day simulation for another quantity production x =1,2,3…8
- Compare the total profit for each one
- Pick the best profit to be the suggested production quantity
- Of course if we run the simulation for more days we get more accurate estimate.

10-days Simulation Results for various production quantities

Production Size	Ten Day Simulated profit $
1	25
2	44
3	60
4	79
5	90
6	93
7	91
8	89
9	5
10	6

From the table it is clear that the best production quantity that maximizes the profit is at x=6. The results are based on only 10-day simulation.

The role of random numbers in simulation

- Suppose we select random numbers in sets of two digits.
- This will provide us with 100 two-digit random numbers from 00 to 99 with each two-digit random number having a 1/100 chance of being selected
- 0 unites, the relative frequency of 0 is 5% Thus we want 5% of the 100 possible two-digit random numbers to correspond to a demand of 0 units.
- While choosing any five numbers of the 100 numbers will do we may assign a demand of 0 to the first 5 numbers i.e. 00, 01,02,03, and 04

Random number Intervals and the daily demand

Daily Dema	relative Frequen	Interval of Random num
0	0.15	00 to 04
1	0.1	05 to 14
2	0.05	15 to 19
3	0.1	20 to 29
4	0.15	30 to 44
5	0.3	45 to 74
6	0.15	75 to 89
7	0.05	90 to 94
8	0.05	95 to 99

Results of simulating ten daily demands

Random number Simulated daily demand	simulated daily demand
63	5
27	3
15	2
99	8
86	6
71	5
74	5
45	5
11	1
2	0

The role of random numbers

- For any simulation problem in which a relative frequency distribution of a variable can be developed,
- It is easy to apply the above random number based procedure to simulate values of the variable.
- First, develop a table of intervals by associating an interval of random numbers with each possible value of the variable
- Then as each random number is selected, you can simply check the corresponding interval and find the associated value of the variable.

The role of random numbers

- Obviously, for long and complex simulations that require numerous calculations, a high speed computer simulation process is desirable.
- In computer simulation pseudo- random numbers are used in exactly the same way as the random numbers selected from random number tables above.
- It would be very risky to make a decision based on the results of such a short period of simulation.

The role of random numbers

- When we think of performing the simulation calculations for a simulated period as long as 500 days, the problems of carrying out the simulation for even a case as small as the bakery shop problem are significant.
- For example let us consider the 500 days. The mathematical model does not change but the work we have to go through to evaluate the results does change but expands . Now we can create a table similar to the ten-day table to evaluate each order size for 500 days of operation.