

Probability

- Probability is a measure of how likely it is for an event to happen.
- We name a probability with a number from 0 to 1.
- If an event is certain to happen, then the probability of the event is 1.
- If an event is certain not to happen, then the probability of the event is 0.

Probability Vs Statistics

- **In probability theory:** R.V. is specified and their parameters are known.
- Goal: Compute probabilities of random values that these variables can take.
- **In statistics:** The values of random variables are known “ from experiment” but theoretical characteristics are unknown.
- Goal: To determine the unknown theoretical characteristics of R.V.
- Probability and Statistics are complementary subjects

What is an Event?

- In probability theory, **an event** is a set of outcomes (a subset of the sample space) to which a probability is assigned.
- Typically, when the sample space is finite, any subset of the sample space is an event (i.e. all elements of the power set of the sample space are defined as events).

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Fundamentals

- We measure the probability for Random Events
 - How likely an event would occur
- The set of all possible events is called Sample Space
- In each experiment, an event may occur with a certain probability (Probability Measure)
- Example:
 - Tossing a dice with 6 faces
 - The sample space is {1, 2, 3, 4, 5, 6}
 - Getting the Event « 2 » in on experiment has a probability $1/6$

Examples

- A single card is pulled (out of 52 cards).
 - **Possible Events**
 - having a red card ($P=1/2$);
 - Having a Jack ($P= 1/13$);
- Two true 6-sided dice are used to consider the event where the sum of the up faces is 10.
 - $P = 3 / 36 = 1/12$

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Probability

- ▶ The **probability** of every set of possible events is **between 0 and 1**, inclusive.
- ▶ The probability of the whole set of outcomes is 1.
 - ▶ Sum of all probability is equal to one
 - ▶ Example for a dice: $P(1)+P(2)+P(3)+ P(4)+P(5)+P(6)=1$
- ▶ If A and B are two events with **no common outcomes**, then the probability of their union is the sum of their probabilities.
 - ▶ Event $E1=\{1\}$,
 - ▶ Event $E2 =\{6\}$
 - ▶ $P(E1 \vee E2)=P(E1)+P(E2)$

Random Variables

An Experiment: is a process whose outcome is not known with certainty

Sample Space: set of outcomes S

Ex: $S = \{H, T\}$, $S = \{1, 2, 3, 4, 5, 6\}$

Random Variable: also known as **stochastic variable**. is a function that assigns a real number to each point in the space

Random Variable is either **discrete** or **continuous**

A random variable: Examples.

- ▶ The waiting time of a customer in a queue
- ▶ The number of cars that enters the parking each hour
- ▶ The number of students that succeed in the exam

Probability Distribution

- ▶ The **probability distribution** of a discrete random variable is a **list of probabilities** associated with each of its possible values.
- ▶ It is also sometimes called the **probability function** or the **probability mass function (PMF)** for discrete random variable.

Probability Mass Function (PMF)

- ▶ The **probability distribution** or **probability mass function (PMF)** of a **discrete random variable** X is a function that gives the probability $p(x_i)$ that the random variable equals some value x_i , for each value x_i :
- ▶ It satisfies the following conditions:

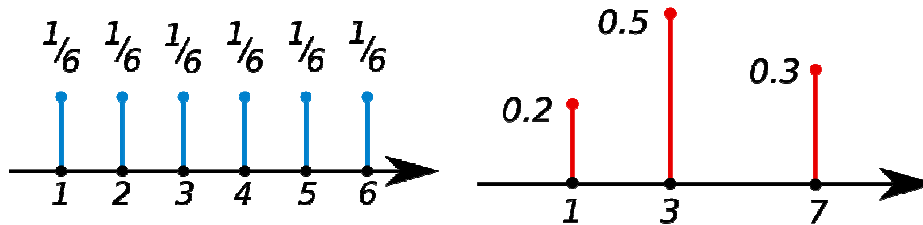
$$p(x_i) = P(X = x_i)$$

$$0 \leq p(x_i) \leq 1$$

$$\sum_i p(x_i) = 1$$

Probability Mass Function

PMF of a fair Dice



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Continuous Random Variable

- ▶ A **continuous random variable** is one which takes an infinite number of possible values.
- ▶ Continuous random variables are usually measurements.
- ▶ Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

Distribution function *aggregates*

- ▶ For the **case of continuous variables**, we do not want to ask what the probability of "1/6" is, because the answer is always 0...
- ▶ Rather, we ask **what is the probability that the value is in the interval (a,b)**.
- ▶ So for continuous variables, we care about the derivative of the distribution function at a point (that's the derivative of an integral). This is called a **probability density function (PDF)**.
- ▶ The probability that a random variable has a value in a set A is the integral of the p.d.f. over that set A.

Probability Density Function (PDF)

- ▶ The **Probability Density Function (PDF)** of a **continuous random variable** is a function that can be integrated to obtain the probability that the random variable takes a value in a given interval.
- ▶ More formally, the probability density function, $f(x)$, of a **continuous random variable** X is the derivative of the cumulative distribution function $F(x)$:

$$f(x) = \frac{d}{dx} F(x)$$

- ▶ Since $F(x) = P(X \leq x)$, it follows that:

$$F(b) - F(a) = P(a \leq X \leq b) = \int_a^b f(x) \cdot dx$$

Cumulative Distribution Function (CDF)

▶ The **Cumulative Distribution Function (CDF)** is a function giving the probability that the random variable X is less than or equal to x , for every value x .

▶ Formally

▶ the cumulative distribution function $F(x)$ is defined to be: $\forall -\infty < x < +\infty$,

$$F(x) = P(X \leq x)$$

Cumulative Distribution Function (CDF)

▶ For a **discrete random variable**, the cumulative distribution function is found by **summing up** the probabilities as in the example below.

$$\forall -\infty < x < +\infty,$$

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i) = \sum_{x_i \leq x} p(x_i)$$

▶ For a **continuous random variable**, the cumulative distribution function is the **integral of its probability density function $f(x)$** .

$$F(a) - F(b) = P(a \leq X \leq b) = \int_a^b f(x) \cdot dx$$

Cumulative Distribution Function (CDF)

► **EX- Discrete case:** Suppose a random variable X has the following probability mass function $p(x_i)$:

x_i	0	1	2	3	4	5
$p(x_i)$	$1/32$	$5/32$	$10/32$	$10/32$	$5/32$	$1/32$

► The cumulative distribution function $F(x)$ is then:

x_i	0	1	2	3	4	5
$F(x_i)$	$1/32$	$6/32$	$16/32$	$26/32$	$31/32$	$32/32$

Discrete Distribution Function

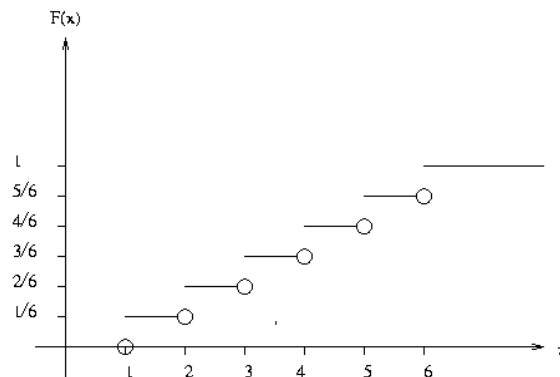


Figure 1.4: Fair die: Graph of the distribution function.

Discrete versus Continuous Random Variables

Discrete Random Variable	Continuous Random Variable
Probability Mass Function (PMF) $p(x_i) = P(X = x_i)$ 1. $p(x_i) \geq 0$, for all i 2. $\sum_{i=1}^{\infty} p(x_i) = 1$	Probability Density Function (PDF) $f(x)$ 1. $f(x) \geq 0$, for all x in R_x 2. $\int_{R_x} f(x) dx = 1$ 3. $f(x) = 0$, if x is not in R_x
Cumulative Distribution Function (CDF) $p(X \leq x)$	
$p(X \leq x) = \sum_{x_i \leq x} p(x_i)$	$p(X \leq x) = \int_{-\infty}^x f(t) dt = 0$ $p(a \leq X \leq b) = \int_a^b f(x) dx$

Mean or Expected Value

Expectation of discrete random variable X

$$\mu_X = E(X) = \sum_{i=1}^n x_i \cdot p(x_i)$$

Expectation of continuous random variable X

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x \cdot f(x) dx$$

Example: Mean and variance

- When a die is thrown, each of the possible faces 1, 2, 3, 4, 5, 6 (the x_i 's) has a probability of $1/6$ (the $p(x_i)$'s) of showing. The expected value of the face showing is therefore:

$$\mu = E(X) = (1 \times 1/6) + (2 \times 1/6) + (3 \times 1/6) + (4 \times 1/6) + (5 \times 1/6) + (6 \times 1/6) = 3.5$$

- Notice that, in this case, $E(X)$ is 3.5, which is not a possible value of X .

Variance

- ▶ The **variance** is a measure of the 'spread' of a distribution about its average value.
- ▶ Variance is symbolized by $V(X)$ or $\text{Var}(X)$ or σ^2 .
 - ▶ The **mean** is a way to describe the location of a distribution,
 - ▶ the **variance** is a way to capture its scale or degree of being spread out. The unit of variance is the square of the unit of the original variable.

Variance

- ▶ The Variance of the random variable X is defined as:

$$V(X) = \sigma_X^2 = E(X - E(X))^2 = E(X^2) - E(X)^2$$

- ▶ where E(X) is the expected value of the random variable X.
- ▶ The **standard deviation** is defined as the square root of the variance, i.e.:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{V(X)} = s$$

Coefficient of Variation

- The Coefficient of Variance of the random variable X is defined as:

$$CV(X) = \frac{V(X)}{E(X)} = \frac{\sigma_X}{\mu_X}$$

- Gives useful information about the distribution. Ex. cv=1 for any exponential distribution regardless of λ . Therefore if we found cv close to 1 in some distribution, we may suggest that it is an exp. distribution

Mean and Variance

$E(X)$ the expected value

Discrete: $E(x) = \sum x_i p(x_i)$

Continuous: $E(x) = \int_0^{\infty} x f(x)$

$Var(x)$ the variance

Discrete $Var(X) = \sum_{i=0}^n (x_i - \mu_i)^2 \cdot p(x_i) = \sum_{i=0}^n x_i^2 \cdot p(x_i) - \left(\sum_{i=0}^n x_i \cdot p(x_i) \right)^2$

Continuous $Var(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 \cdot f(x) dx = \left(\int_{-\infty}^{\infty} x^2 \cdot f(x) dx \right) - \mu_x^2$

Discrete Probability Distribution

- Bernoulli Trials
- Binomial Distribution
- Geometric Distribution
- Poisson Distribution
- Poisson Process

Bernoulli Trials

Any simple trial with two possible outcomes. p and q

EX: Tossing a coin, repeat, with counting # of success p
“the number of heads”

Then # of failure $q=(1-p)$, *“the number of tails”*

$P(\text{HHT}) = p \cdot p \cdot q$

$P(\text{TTT}) = q \cdot q \cdot q$

If we have k as the number of successes and $n-k$ failures

Then the probability is $p^k q^{n-k}$

Binomial Random Variable

If we have $X : S \rightarrow \{0,1,2,3\}$

Where X is the number of successes

$$X(sss) = 3$$

$$X(sfs) = X(ssf) = 2 \quad \dots \text{etc}$$

X now is a random variable.

X is named Binomial random variable resulted from n Bernoulli trials denoted: $b(n, p)$

Modeling of Random Events with Two-States

Binomial Random Variable

Now the probability that $X = k$ $0 \leq k \leq n$
that is all strings with k success and $n - k$ fails,
there are $\binom{n}{k}$ different ways

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

Remark: $\sum_{k=0}^n \binom{n}{k} p^k q^{n-k} = (p + q)^n = 1$

Geometric Random Variable

- Consider independent Bernoulli trials are performed until success $s, fs, ffs, fffs, \dots$

$$P(X = n) = P(fff \dots fs) = q^{n-1} p = pq^{n-1}$$

- Remark: $\sum_{n=1}^{\infty} pq^{n-1} = p(1 + q + q^2 + \dots) = p \frac{1}{1-q} = 1$
- Exr: For a geometric variable X compute $P(X > k)$

Geometric Random Variable

$$\text{PMF: } p(X = k) = \begin{cases} q^{k-1}p, & k = 0, 1, 2, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F(X) = p(X \leq k) = 1 - (1-p)^k$$

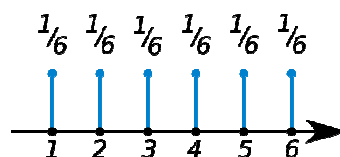
$$\text{Expected Value: } E[X] = \frac{1}{p}$$

$$\text{Variance: } V[X] = \sigma^2 = \frac{q}{p^2} = \frac{1-p}{p^2}$$

Uniform Random Variable

- An R.V. Takes values $1, 2, 3, \dots, n$ with equal probabilities

$$P(X = k) = \frac{1}{n}$$



Poisson

- For $X = b(n, p)$ with large n and small p , it is useless to compute the exact $P(X = k)$, as it involves huge calculations of factorial n

$$P(X = k) = \binom{n}{k} p^k q^{n-k} = \frac{n(n-1)\dots(n-k+1)}{\text{fact}(k)} p^k q^{n-k}$$

- For large n , $n-k+1$ is approximated to n

$$P(X = k) \approx \frac{n^k}{k!} p^k (1-p)^n = \frac{(np)^k}{k!} \left[(1-p)^{\frac{1}{p}} \right]^{np}$$

$$\lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}} = e^{-1}, \text{ small } x$$

$$P(X = k) = \frac{(np)^k}{k!} e^{-np}$$

$$P(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda}, \lambda = np$$

Poisson

- Ex. A production line with .4 percent of its items are defective, $n=500$ items are taken for a quality control. What is the probability that 0, 1, 3 items of them are defective
- That is $X=b(500,.004)$ aprox. To Poisson

$$P(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda}$$

$$\lambda = 500 * .004 = 2$$

$$P(x = 0) = e^{-2}$$

$$P(x = 1) = 2e^{-2}$$

$$P(x = 3) = \frac{4}{3} e^{-2}$$

Example: Poisson Distribution

- The number of cars that enter the parking follows a Poisson distribution with a mean rate equal to $\lambda = 20$ **cars/hour**
 - The probability of having **exactly 15 cars** entering the parking in **one hour**:

$$p(15) = P(X = 15) = \frac{20^{15}}{15!} \cdot \exp(-20) = 0.051649$$

Applications of Poisson

- ▶ **Context:** number of events occurring in a fixed period of time
 - ▶ Events occur with a known average rate and are **independent**
 - ▶ Poisson distribution is characterized by the **average rate** λ
 - ▶ The average number of arrival in the fixed time period.
- ▶ **Examples**
 - ▶ The number of cars passing a fixed point in a 5 minute interval. **Average rate: $\lambda = 3$ cars/5 minutes**
 - ▶ The number of calls received by a switchboard during a given period of time. **Average rate: $\lambda = 3$ call/minutes**
 - ▶ The number of message coming to a router per second
 - ▶ The number of travelers arriving to the airport for flight registration

Poisson Distribution

- The Poisson distribution with the average rate parameter λ

$$\text{PMF: } p(k) = P(X = k) = \begin{cases} \frac{\lambda^k}{k!} \exp(-\lambda) & \text{for } k = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F(k) = P(X \leq k) = \sum_{i=0}^k \frac{\lambda^i}{i!} \cdot \exp(-\lambda)$$

$$\text{Expected value: } E[X] = \lambda$$

$$\text{Variance: } V[X] = \lambda$$

Continuous Probability Distribution

- uniform Distribution
- exponential Distribution
- Normal Distribution
- Standard Normal Process

Continuous Uniform Distribution

- The **continuous uniform distribution** is a family of probability distributions such that for each member of the family, all intervals of the same length on the distribution's support are **equally probable**
- A random variable X is **uniformly distributed** on the interval $[a,b]$, $U(a,b)$, if its PDF and CDF are:

$$\text{PDF: } f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

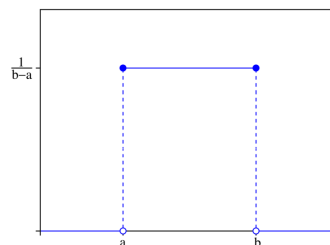
$$\text{Expected value: } E[X] = \frac{a+b}{2}$$

$$\text{Variance: } V[X] = \frac{(a+b)^2}{12}$$

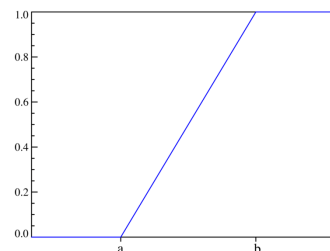
Uniform Distribution U(a,b)

- The PDF is $f(x) = \text{const} = \frac{1}{b-a}$
- Properties**
 - $p(x_1 \leq X \leq x_2)$ is proportional to the length of the interval

$$F(x_2) - F(x_1) = \frac{x_2 - x_1}{b-a}$$



- Special case:** a **standard uniform distribution** $U(0,1)$.
 - Very useful for random number generators in simulators

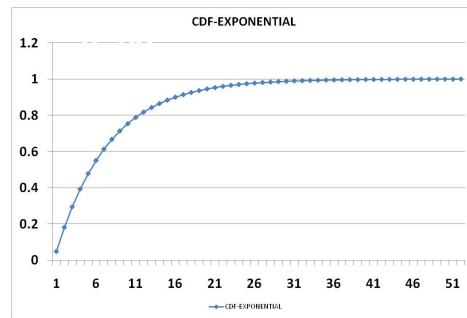
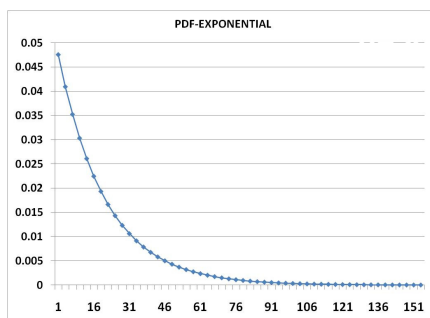


Exponential Distribution

$$\text{PDF: } f(x) = \begin{cases} \lambda \cdot \exp(-\lambda \cdot x), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Exponential Distribution



$$f(x) = \begin{cases} \frac{1}{20} \cdot \exp\left(-\frac{x}{20}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{20}\right), & x \geq 0 \end{cases}$$

Exponential Distribution

- (Special interest) The **exponential distribution** describes the times between events in a Poisson process, in which events occur continuously and independently at a constant average rate.

- A random variable X is **exponentially distributed** with parameter $\mu=1/\lambda > 0$ if its PDF and CDF are:

$$\text{PDF: } f(x) = \begin{cases} \lambda \cdot \exp(-\lambda \cdot x), & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad \longrightarrow \quad f(x) = \begin{cases} \frac{1}{\mu} \cdot \exp\left(-\frac{x}{\mu}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF: } F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases} \quad \longrightarrow \quad F(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{\mu}\right), & x \geq 0 \end{cases}$$

$$\text{Expected value: } E[X] = \frac{1}{\lambda} = \mu$$

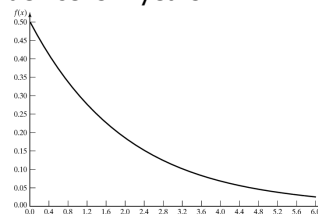
$$\text{Variance: } V[X] = \frac{1}{\lambda^2} = \mu^2$$

Example: Continuous Random Variables

Ex.: modeling the lifetime of a device

- Time is a **continuous random variable**
 - Random Time is typically modeled as **exponential distribution**
 - We assume that with **average** lifetime of a device is 2 years

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



- Probability that the device's life is between 2 and 3 years is:

$$P(2 \leq x \leq 3) = \frac{1}{2} \int_2^3 e^{-x/2} dx = 0.14$$

The life time Ex.

- **Cumulative Distribution Function:** A device has the CDF:

$$F(x) = \frac{1}{2} \int_0^x e^{-t/2} dt = 1 - e^{-x/2}$$

- The probability that the device lasts for less than 2 years:

$$P(0 \leq X \leq 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

- The probability that it lasts between 2 and 3 years:

$$P(2 \leq X \leq 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

The life time Ex.

Expected Value and Variance

- Example: The mean of life of the previous device is:

$$E(X) = \frac{1}{2} \int_0^{\infty} x e^{-x/2} dx = -x e^{-x/2} \Big|_0^{\infty} + \int_0^{\infty} e^{-x/2} dx = 2$$

- To compute variance of X , we first compute $E(X^2)$:

$$E(X^2) = \frac{1}{2} \int_0^{\infty} x^2 e^{-x/2} dx = -x^2 e^{-x/2} \Big|_0^{\infty} + \int_0^{\infty} e^{-x/2} dx = 8$$

- Hence, the variance and standard deviation of the device's life are:

$$V(X) = 8 - 2^2 = 4$$

$$\sigma = \sqrt{V(X)} = 2$$

Exponential Distribution

- ▶ **The memoryless property:** In probability theory, **memoryless** is a property of certain probability distributions: the **exponential distributions** and the **geometric distributions**, wherein any derived probability from a set of random samples is distinct and has no information (i.e. "memory") of earlier samples.

- ▶ Formally, the **memoryless property** is:

For all s and t greater or equal to 0:

$$p(X > s + t | X > s) = p(X > t)$$

- ▶ This means that the **future event** do not depend on the **past event**, but only on the **present event**

Normal Distribution

- The **Normal distribution**, also called the **Gaussian distribution**, is an important family of continuous probability distributions, applicable in many fields.
- Each member of the family may be defined by two parameters, **location** and **scale: the mean** ("average", μ) and **variance** (standard deviation squared, σ^2) respectively.
- The importance of the normal distribution as a model of quantitative phenomena in the **natural** and **behavioral** sciences is due in part to the **Central Limit Theorem**.
- It is usually used to model system error (e.g. channel error), the distribution of natural phenomena, height, weight, etc.

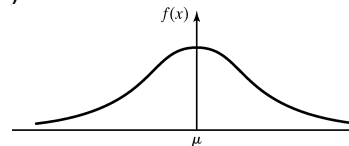
Normal or Gaussian Distribution

- A **continuous random variable** X , taking all real values in the range $(-\infty, +\infty)$ is said to follow a **Normal distribution** with parameters μ and σ if it has the following PDF and CDF:

$$\text{PDF: } f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

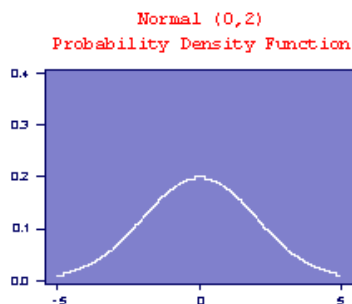
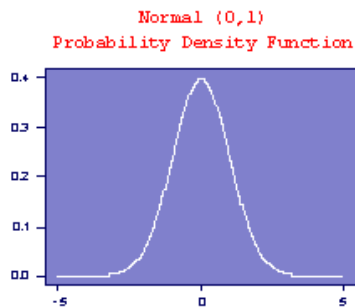
$$\text{CDF: } F(x) = \frac{1}{2} \cdot \left(1 + \text{erf}\left(\frac{x-\mu}{\sigma \cdot \sqrt{2}}\right)\right) \quad \text{where Error Function: } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x \exp(-t^2)$$

- The Normal distribution is denoted as $X \sim N(\mu, \sigma^2)$
- This probability density function (PDF) is
 - a symmetrical, bell-shaped curve,
 - centered at its expected value μ .
 - The variance is σ^2 .



Normal distribution

- Example**
- The simplest case of the normal distribution, known as the **Standard Normal Distribution**, has expected value zero and variance one. This is written as $N(0,1)$.



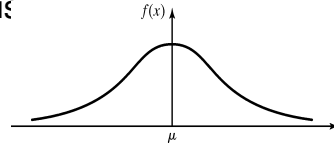
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 - a symmetrical, bell-shaped curve,
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Standard Normal Distribution

Independent of μ and σ , using the **standard normal distribution**:

- Transformation of variables: let

$$Z \sim N(0,1)$$

$$Z = \frac{X - \mu}{\sigma}$$

$$F(x) = P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

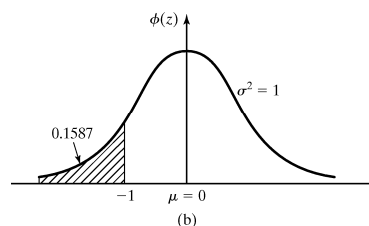
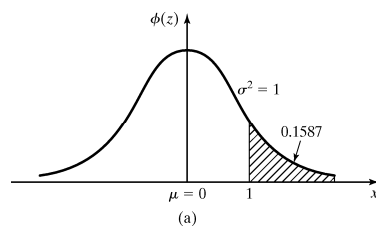
$$\text{, where } \Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Standard Normal Distribution

- Note that $f_Z(x)$ *is positive for all* $-\infty < x < \infty$, *hence Z takes on all real values*, its range is the entire real line. Also note that $f_Z(x)$ *is an even function*
- The graph of $f_Z(x)$ *is a bell-shaped curve, symmetric about the y-axis.*
- This curve is called a gaussian curve. Its maximum is attained at $x = 0$, then it decreases on both sides of its top point. Actually, it decreases very fast.

Normal Distribution

- Example: The time required to load a transporting truck, X , is distributed as $N(12,4)$
 - The probability that the truck is loaded in less than 10 hours:
$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = 0.1587$$
 - Using the symmetry property, $\Phi(1)$ is the complement of $\Phi(-1)$



Empirical Distributions

- An Empirical Distribution is a distribution whose parameters are the observed values in a sample of data.
 - May be used when it is impossible or unnecessary to establish that a random variable has any particular parametric distribution.
 - **Advantage**: no assumption beyond the observed values in the sample.
 - **Disadvantage**: sample might not cover the entire range of possible values.

Empirical Distributions

- ▶ In statistics, an **empirical distribution function** is a **cumulative probability distribution function** that concentrates probability $1/n$ at each of the n numbers in a sample.
- ▶ Let x_1, \dots, x_n be iid random variables in with the CDF equal to $F(x)$.
- ▶ The **empirical distribution function** $F_n(x)$ based on sample x_1, \dots, x_n is a step function defined by

$$F_n(x) = \frac{\text{number of element in the sample } \leq x}{n} = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x)$$

where $I(\mathcal{A})$ is the **indicator of event** \mathcal{A} . $I(X_i \leq x) = \begin{cases} 1 & \text{if } (X_i \leq x) \\ 0 & \text{otherwise} \end{cases}$

- ▶ For a fixed value x , $I(X_i \leq x)$ is a **Bernoulli (Trial)** random variable with parameter $p=F(x)$, hence $nF_n(x)$ is a **binomial** random variable with mean $nF(x)$ and variance $nF(x)(1-F(x))$.