Functions of Random Variables

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$$0 < X < 1$$
$$-1 < -X < 0$$
$$0 < 1-X < 1$$
$$-\infty < \ln(1-X) < 0$$
$$0 < Y < \infty$$

$$F_Y(y) = P(Y \le y) = P(\frac{-1}{\lambda}\ln(1-X) \le y)$$
$$= P(1-X \ge e^{-\lambda y}) = P(X \le 1-e^{-\lambda y})$$

$$F_Y(y) = 1-e^{-\lambda y}$$
$$f_Y(y) = \lambda e^{-\lambda y}$$

Mean Variance

Properties of the mean:

$$1)E[cX]=cE[X]$$

$$2)E[c_1X_1+c_2X_2]=c_1E[X_1]+c_2E[X_2]$$

Properties of the variance:

$$1)Var(X)\geq 0$$

$$2)Var(cX)=c^2Var(X)$$

$$3)Var(\sum_{i=1}^{n}X_i)=\sum_{i=1}^{n}Var(X_i)$$

Rules

- Exr. $1)X=c\Rightarrow Var(X)=0$

  $2)Y=aX\Rightarrow Var(Y)=a^2Var(X)$

  $3)Y=X+b\Rightarrow Var(Y)=Var(X)$

  $4)Y=X_1+X_2\Rightarrow Var(Y)=Var(X_1)+Var(X_2)$

- Question Is

$$Var(X-Y)=Var(X)-Var(Y)$$

Example

Let X and Y be independent and

E(X)= 5, E(Y)=-3, and $\sigma_X = 2, \sigma_Y = 3$

Find the mean and he Std Deviation of

Z=3X-2Y-2

E[Z]= 3E[X]-2E[Y]-2 = 15+6-2=19

Var(Z)= 9Var(X)+4Var(Y) = 9.4+4.9 = 72

$\sigma_Z = \sqrt{72}$

Covariance

- The covariance is a measure of dependency between two variables
- Def. Cov(X,Y)= E[(X-E[X]) * (Y-E[Y])

= E[X,Y]-E[X]*E[Y]

For dependent Variables X, Y

Var(X+Y)=Var(X)+Var(Y) + 2E[X,Y] – 2E[X].E[Y]

= Var(X)+Var(Y)+2Cov(X,Y)

Remark: Large Covariance → Large dependency

But If the covariance is 0, it does not mean they are independent.

Rules

(Prove)

1. Cov(X,X)=Var(X)
2. Cov(X,Y)=Cov(Y,X)
3. Cov. Is linear i.e.

Cov(aX+bY,Z)=aCov(X,Z)+bCov(Y,Z)

Exr. For X,Y independent U(0,1)

Find $Cov(X+2Y, X^2 - Y)$

Correlation Analysis

- In probability theory and statistics, **correlation** (often measured as a correlation coefficient) indicates **the strength and direction of a linear relationship between two random variables**.
- **Correlation** refers to the departure of two variables from independence. In this broad sense there are several coefficients, measuring the degree of correlation, adapted to the nature of the data.

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y},$$

Correlation Analysis

- Remark: $-1 \le \rho_{(X,Y)} \le 1$
- The sign of ρ indicates the direction of the relationship;
  - ρ near 0 indicates no linear relationship,
  - ρ near 1 or -1 indicates a strong linear relationship.

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- Example: For X= U(0,1) find $\rho_{(X,X^2)}$

$Var(X) = \frac{1}{12}$ and $Var(X^2) = E[X^4] - (E[X^2])^2 = \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}$

$$\rho(X,X^2) = \frac{Cov(X,X^2)}{\sigma_X \sigma_{X^2}} = \frac{1/12}{\sqrt{1/12}\sqrt{4/45}} \approx 0.996$$

i.e. very strong dependency

Joint Probability

- **Joint Probability (A ∩ B)**
  - The probability of two events in conjunction. It is the probability of both events together.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

- **Independent Events**
  - Two events A and B are **independent** if

$$p(A \cap B) = p(A) \cdot p(B)$$

Joint Distribution

- For two random variables X and Y

$$F_{(X,Y)}(x, y) = p\{X \le x, Y \le y\}$$

- Example: For X,Y ind. U(0,1) r.v.

$$F_{(X,Y)}(x, y) = P(X \le x, Y \le y) = P(X \le x).P(Y \le y)$$
$$= F_X(x).F_Y(y) = x.y$$
$$f_{(X,Y)}(x, y) = \frac{\partial^2 F_{(X,Y)}(x, y)}{\partial x \partial y} = 1$$

Joint Probability density function

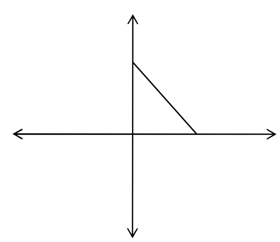
- X,Y are called jointly continuous if there exists f(x,y)
- S.t. $P(X \in A, y \in B) = \int_B \int_A f(xy)dxdy$
- If X, Y are independent then

$$f(x, y) = f_X(x) f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Example



- Find the Cor(X,Y) for

$$f(x, y) = \begin{cases} 24xy & x \geq 0, y \geq 0 \quad x + y \leq 1 \\ o & o.w. \end{cases}$$

Find $Cov(X, Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)dydx = \int_0^1 \int_0^{1-x} xyf(x, y)dydx$$

$$= \int_0^1 \int_0^1 xy(24xy)dydx = \int_0^1 24x^2 \int_0^{1-x} y^2 dydx$$

$$= \int_0^1 24x^2 \frac{1}{3} y^3 |_0^{1-x} dx = \int_0^1 8x^2(1-x)^3 dx = \frac{2}{15}$$

Computing the covariance

$$E[X] = \int_1^1 x f_x(x)dx$$

$$f_x(x) = \int_1^{1-x} 24xydy = 12xy^2 |_0^{1-x} = 12x(1-x)^2$$

$$f_y(y) = \int_1^{1-y} 24xydx = 12x^2 y |_0^{1-y} = 12y(1-y)^2$$

$$E[x] = \int_0^1 12x(1-x)^2 dx = \frac{2}{5}$$

$$E[y] = \int_0^1 12y(1-y)^2 dy = \frac{2}{5}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{15} - \frac{2}{5}.\frac{2}{5} = \frac{-2}{75}$$

Computing the Correlation

$$E[X^2] = \int_0^1 x^2 f_x(x)dx = \int_0^1 x^2 12x(1-x)^2 dx$$

$$= \int_0^1 12x^3(1-2x+x^2)dx = 4x^4 - \frac{24}{5}x^5 + \frac{12}{6}x^6 |_0^1$$

$$= 3 - \frac{24}{5} + \frac{12}{6}$$

$$Var(X) = E[X^2] - (E[X])^2 = 3 - \frac{24}{5} + \frac{12}{6} - \frac{4}{25} = \frac{1}{25}$$

Similarly $Var(X) = \frac{1}{25}$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-2/75}{1/25} = \frac{-2}{3}$$