Discrete Probability Distribution Review

- Bernoulli Trials
- Binomial Distribution
- Geometric Distribution
- Poisson Distribution
- Poisson Process

Siméon Denis Poisson

(1781-1840) France

"Life is good for only two things: discovering mathematics and teaching mathematics."

Applications of Poisson Review

- **Context:** number of events occurring in a fixed period of time
  - Events occur with a known average rate and are independent
- Possion distribution is characterized by the **average rate** λ
  - The average number of arrival in the fixed time period.

- **Examples**
  - The number of cars passing a fixed point in a 5 minute interval. Average rate: λ = 3 cars/5 minutes
  - The number of calls received by a switchboard during a given period of time. Average rate: λ =3 call/minutes
  - The number of message coming to a router per second
  - The number of travelers arriving to the airport for flight registration

Poisson Distribution Review

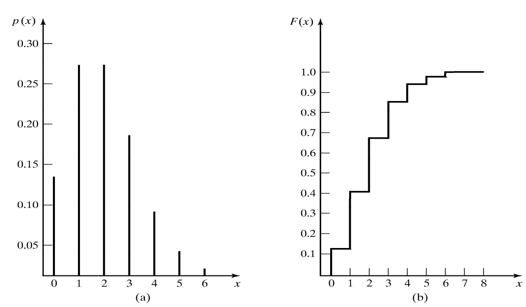
- The Poisson distribution with the average rate parameter λ

$$\text{PMF:} \quad p(k) = P(X = k) = \begin{cases} \dfrac{\lambda^k}{k!}\exp(-\lambda) & \text{for } k = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF:} \quad F(k) = p(X \le k) = \sum_{i=0}^{k} \frac{\lambda^i}{i!} \cdot \exp(-\lambda)$$

Expected value: $E[X] = \lambda$

Variance: $V[X] = \lambda$



Poisson R.V. Review

- Ex. A production line with .4 percent of its items are defective , n=500 items are taken for a quality control. What is the probability that 0, 1, 3 items of them are defective
- That is X=b(500,0.004) aprox. To Poisson

$$P(X=k)=\frac{(\lambda)^k}{k!}e^{-\lambda}$$

$$\lambda = 500 * .004 = 2$$

$$P(x=0)=e^{-2}$$

$$P(x=1)=2e^{-2}$$

$$P(x=3)=\frac{4}{3}e^{-2}$$

Example: Poisson Review

- The number of cars that enter the parking follows a Poisson distribution with a mean rate equal to λ = 20 **cars/hour**
  - The probability of having **exactly** **15 cars** entering the parking in **one hour**:

$$p(15)=P(X=15)=\frac{20^{15}}{15!}\cdot \exp(-20)=0.051649$$

Poisson Distribution [Discrete]

- Example: A computer repair person is "beeped" each time there is a call for service. The number of beeps per hour ~ Poisson(α = 2 per hour).
  - The probability of three beeps in the next hour:

$$P(X=3)=\frac{(\lambda)^k}{k!}e^{-\lambda}=\frac{(2)^3}{3!}e^{-2}=0.18$$

$$Or$$

$$P(X=3)=F(3)-F(2)=857-.677=0.18$$

  - The probability of two or more beeps in a 1-hour period:

$$P(X>=2)=1-p(0)-p(1)=1-F(1)=0.594$$

Poisson

- Mass function $p(x)=\begin{cases}\dfrac{e^{-\lambda}\lambda^x}{x!} & x\in\{0,1,\ldots,t\}\\ 0 & \text{otherwise}\end{cases}$
- Algorithm

  1. Let $a=e^{-\lambda}, b=1, i=0$
  2. Generate $U_{i+1}\sim U(0,1)$ and replace b by bU_{i+1}. If $b<a$, return $X=i$. Otherwise go to step 3.
  3. Let $i=i+1$ and go back to step 1

- Rather slow. No very good algorithm for Poisson distribution

Poisson Process

- A stochastic process $\{N(t), t \leq 0\}$ that counts the number of events until time t
- ***Examples:***
  - The arrival of "customers" is commonly modeled as a Poisson process in the study of simple queuing systems.
  - The number of web pages requests arriving at a server (except for unusual circumstances such as coordinated denial of service attacks.)

Stationary & Independent Increments

- A counting process has ***independent increments*** if, for any

  $0 \leq s < t \leq u < v,\ N(t) - N(s)$ is independent of $N(v) - N(u)$

  That is, the numbers of events that occur in nonoverlapping intervals are independent random variables.

- A counting process has ***stationary increments*** if the distribution if, for any $s < t$, the distribution of $N(t) - N(s)$ depends only on the length of the time interval, $t - s$.

Stationary Poisson Process

- ***Also known as homogeneous* Poisson process** is characterized by a **CONSTANT rate parameter λ**, also known as **intensity**, such that the number of events in time interval $(t, t+\tau]$ follows a Poisson distribution with associated parameter $\lambda \cdot \tau$.
- Formally, A counting process {***N***(***t***), ***t***≤ 0} is a (Stationary) Poisson process with mean rate λ if:

  for $t \geq 0$ and $n = 0, 1, 2, \ldots$

  PMF: $p\left[N(t+\tau) - N(t) = n\right] = p\left[N(\tau) = n\right] = \dfrac{(\lambda \cdot \tau)^n}{n!} \exp(-\lambda \cdot \tau)$
- $N(t+\tau) - N(t)$ describes the number of events in time interval $(t, t+\tau]$
- The mean and the variance are equal $E\left[N(\tau)\right] = V\left[N(\tau)\right] = \lambda \cdot \tau$

Stationary Poisson Process

- **Properties of Poisson process**
  - Arrivals occur one at a time (not simultaneous)
  - $\{N(\tau), \tau \geq 0\}$ has **stationary increments**, which means $N(t) - N(s) = N(t-s)$

  The number of arrivals in time s to t is also Poisson-distributed with mean $\lambda \cdot (t-s)$

  - $\{N(\tau), \tau \geq 0\}$ has **independent increments**

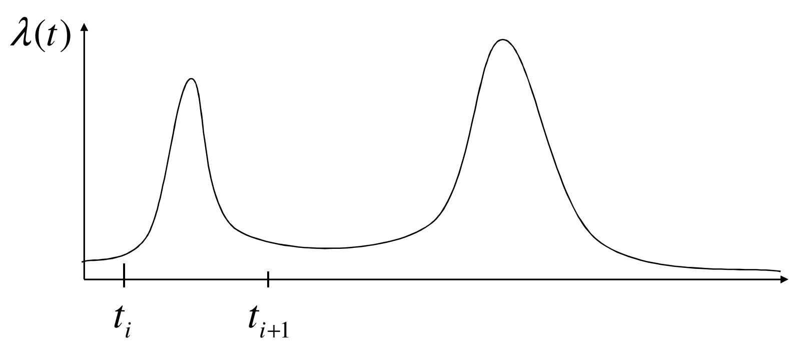
Generating a Poisson Process

- Stationary with rate λ>0
- Time between events $A_i = t_i - t_{i-1}$ are IID exponential
- Algorithm

1. Generate $U \sim U(0,1)$
2. Return $t_i = t_{i-1} - \left(1/\lambda\right)\ln U$

Nonstationary Case

- Can we simply generalize?



Thinning Algorithm

1. Set $t = t_{i-1}$
2. Generate U_1, U_2 I.I.D. U(0,1)
3. Replace t by

$$t - \left(\frac{1}{\lambda^*}\right)\ln U_1, \text{ where } \lambda^* = \max_t\{\lambda(t)\}$$

4. If $U_2 \le \lambda(t)/\lambda^*$ return $t_i = t$. Otherwise, go back to step 2.