

Quiz on coming Thur. on Generating Random Variates

Convolution

- For some distributions, the desired R.V. X can be expressed as sum of other random variables that are IID with the same CDF
- Assume that $X = Y_1 + Y_2 + \dots + Y_m$
(where the Y s are IID with the same CDF)
- Algorithm
 1. Generate Y_1, Y_2, \dots, Y_m IID each with CDF
 2. Return $X = Y_1 + Y_2 + \dots + Y_m$

Erlang- m Distribution

The Erlang- m random variable with mean β , then

$$X = Y_1 + Y_2 + \dots + Y_m ,$$

where the Y_i s are IID exponential random variables, each with mean β/m . If we use the inverse-transform method to generate the exponential Y_i s, then

$$X = \sum_{i=1}^m Y_i = \sum_{i=1}^m \frac{-\beta}{m} \ln U_i = \frac{-\beta}{m} \ln \left(\prod_{i=1}^m U_i \right).$$

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Erlang- m Distribution

Hence we generate random variate x as follows:

1. Generate u_1, u_2, \dots, u_m as IID $\mathcal{U}(0,1)$.

2. Return $x = \frac{-\beta}{m} \ln \left(\prod_{i=1}^m u_i \right)$.

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Normal Distribution

- Note that if $X \sim N(0,1)$
 $\Rightarrow \mu + \sigma X \sim N(\mu, \sigma)$
- If we can generate unit normal, then we can generate any normal

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

$$\phi(x) = F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{x^2}{2}} dx$$

- Neither the distribution function nor the density function is invertible
- Use indirect method

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Normal: Box-Muller

- Algorithm
 1. Generate independent $U_1, U_2 \sim U(0,1)$
 2. Set $X_1 = \sqrt{-2 \ln U_1} \cos 2\pi U_2$,
 $X_2 = \sqrt{-2 \ln U_1} \sin 2\pi U_2$
 3. Return X_1
- Each one X_1 or X_2 may be used. Each one of them is an i.i.d $N(0,1)$
- Technically, independent $N(0,1)$, but serious problem if used with LCGs

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Normal: Box-Muller

- For any normal distribution $N(\mu, \sigma^2)$
- Generate $S_i = \sigma X_i + \mu$

Acceptance-Rejection Method

By Von Neumann (1951), used when the direct approaches fail or inefficient.

A closed form for $F(X)$ does not exist, so what we'll do is to add another distribution . For which we know "how to calculate the CDF and its inverse".

We pick a function $t(x)$ that is larger than $f(x)$ for all x . Technically we say that $t(x)$ *majorizes* $f(x)$.

Acceptance-Rejection Method

- Specify a function that majorizes the density

$$t(x) \geq f(x), \forall x$$

- ***$t(x)$ is clearly not a density fun.***

- New density function $r(x) = \frac{t(x)}{\int_{-\infty}^{\infty} t(x) dx}$

- Algorithm to generate a r.v. with density ~~$f(x)$~~

1. Generate Y with density r
2. Generate U independent of Y
3. If $U \leq f(Y)/t(Y)$, return $X = Y$.

Otherwise go back to Step 1.

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Beta Distribution

- Density

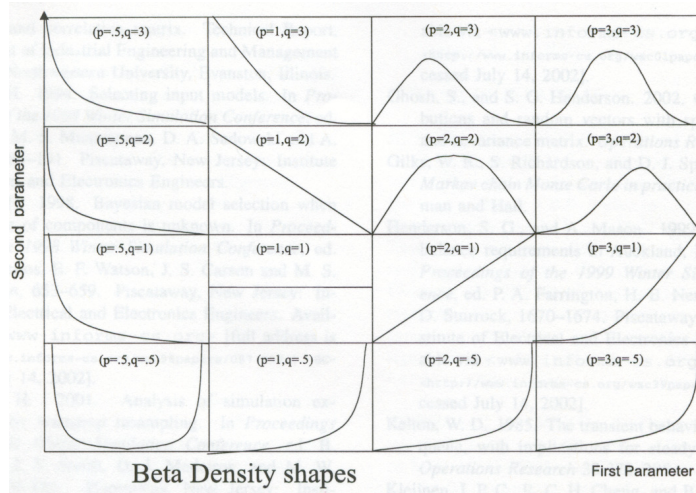
$$f(x) = \begin{cases} \frac{x^{\alpha_1-1} (1-x)^{\alpha_2-1}}{B(\alpha_1, \alpha_2)} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1-1} (1-t)^{\alpha_2-1} dt$$

- No closed form CDF. No closed form inverse
- Must use numerical methods for inverse-transform method

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Beta Distribution Shapes



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Beta(4,3) Example

- Density $f(x) = \begin{cases} 60x^3(1-x)^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$
- The distribution $F(x)$ is a 6th degree polynomial, The inverse approach is not a good one.
- Try to find $t(x)$
 put $df/dx = 0$ to find the max of the f , which will be a $x = 0.6$, and $f(0.6) = 2.0736$
- Put $t(x) = \begin{cases} 2.0736 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$
- Then $r(x) = \frac{t(x)}{\int_{-\infty}^{\infty} t(x) dx} = \frac{2.0736}{\int_0^1 2.0736 dx} = 1$

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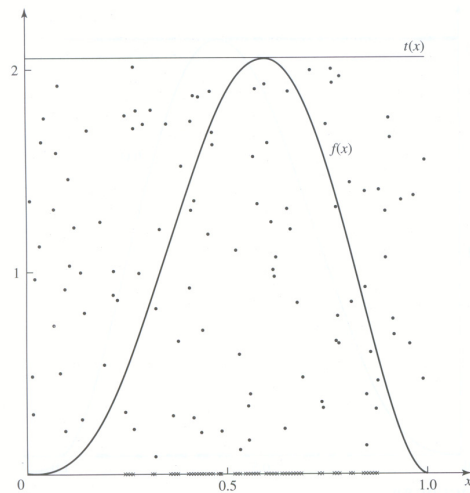
Beta(4,3) Example

So $r(x)$ is just $U(0,1)$

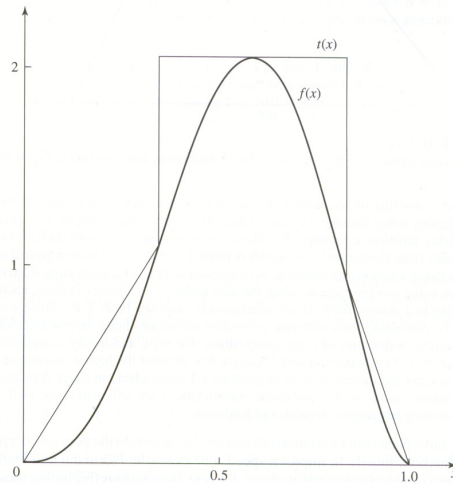
The algorithm:

1. *Generate Y having density of $r(x)$; i.e. $U(0,1)$*
2. *Generate $U \sim U(0,1)$, independent of Y*
3. *If $U \leq \frac{f(Y)}{t(Y)}$, return $X=Y$; Otherwise go to step 1*

Example:



Example: More Efficient



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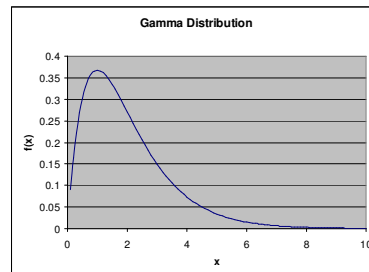
Acceptance – Rejection Technique for Gamma Dist.

A gamma(α, β) density

$$f(x) = \begin{cases} \frac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)} & 0 < x \\ 0 & \text{otherwise} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

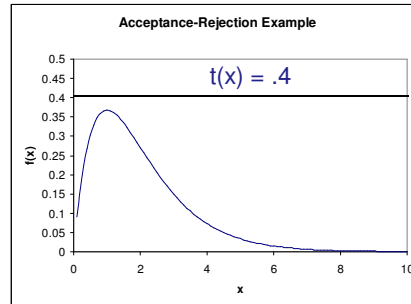


A closed form for $F(X)$
does not exist,

Acceptance – Rejection Technique for Gamma Dist.

A gamma(2,1)

We pick a function $t(x)$ that
is larger than $f(x)$ for all x .
 $t(x)=0.4$



Acceptance-Rejection Gamma(2,1)

In we selected $t(x) = .4$ $0 \leq x \leq 10$

Now $t(x)$ is not a density function, why?

since its integral from 0 to 10 doesn't add up to 1.

So let us define c : And $r(x) = t(x)/c$:

$$c = \int_0^{\infty} t(x) dx$$

$$= \int_0^{10} .4 dx$$

$$= \left|_0^{10} .4x \right.$$

$$= 4$$

$$r(x) = .4/4$$

$$= .1$$

$$R(x) = \int_0^x .1 dx'$$

$$= .1x'$$

i.e. $r(x)$ is a density fn.

Acceptance-Rejection

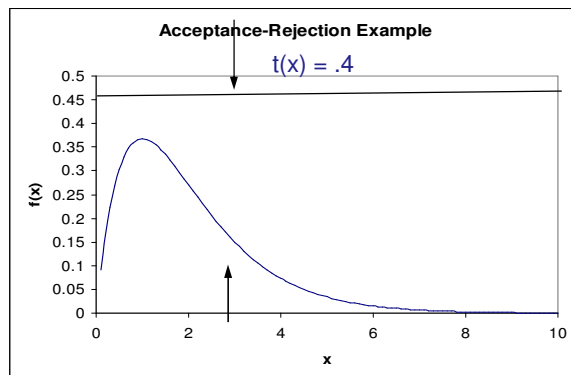
And in this simple case, we can easily determine the inverse transformation for R: $X = 10Y$.

So, let's say we pick a random number $Y = .3$.
This translates into an X of 3

Acceptance – Rejection

The inverse transformation for R: $X = 10Y$.

For a random number $Y = 0.3$,
This translates into an X of 3



If we threw darts that could land only on the line $x = 3$, then the probability that a dart hitting inside the distribution would be $f(X=3)/t(X=3)$.

Acceptance – Rejection

$$\begin{aligned} f(X=.3)/t(X=.3) \\ &= .15/.4 \\ &= .375 \end{aligned}$$

Generate $U \sim U(0,1)$.

If U is less than $.375$, we will accept $X=3$ as coming from a $\text{gamma}(2,1)$ distribution.

Otherwise, we will start the process over by selecting a new R and new U .

