Bernoulli

- Mass function

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

- Algorithm

1. Generate $U \sim U(0,1)$
2. If $U \le p$ return $X = 1$. Otherwise return $X = 0$

Binomial

- Mass function

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in \{0,1,\ldots,n\} \\ 0 & \text{otherwise} \end{cases}$$

- Use the fact that if ***X***~B(***n,p***) then

$$X = Y_1 + Y_2 + \ldots, Y_n$$
$$Y_i \sim \text{Bernoulli}(p)$$

Binomial

If X is a binomial random variable with parameters n and p. Note the sum of n IID Bernoulli(p) random variables is a B(n, p) random variable.

To generate a binomial random variate,

1. Generate $y_1, y_2, \ldots, y_n$ as IID Bernoulli(p) random variates.
2. Set $x = y_1 + y_2 + \ldots + y_n$.

Geometric

- Mass function

$$p(x) = \begin{cases} p(1-p)^x & x \in \{0,1,\ldots,t\} \\ 0 & \text{otherwise} \end{cases}$$

$$C.D.F : F(x) = p(X \leq x) = 1-(1-p)^x$$

- Use inverse-transform

$$1.\ \text{Generate}\ U \sim U(0,1)$$

$$2.\ \text{Return}\ X = \left\lfloor \ln(1-U) \big/ \ln(1-p) \right\rfloor$$

Empirical Discrete Distribution

- Find the random variate for the following discrete distribution.

x	p(x)
0	0.50
1	0.30
2	0.20

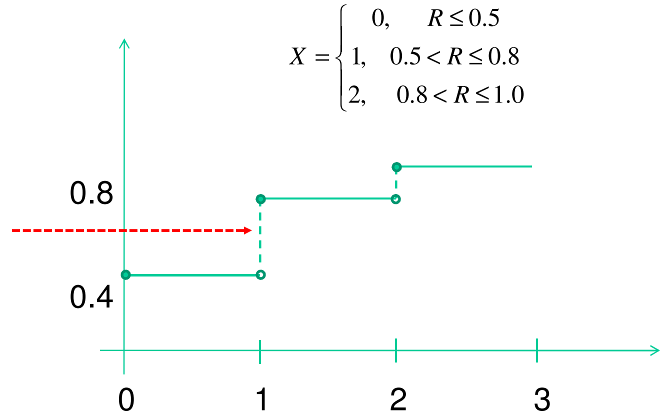
- At first find the F

x	p(x)	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

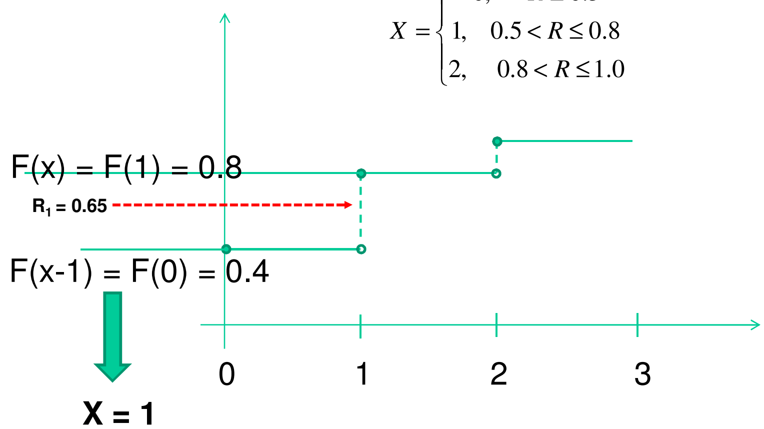
Empirical Discrete Distribution

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$



Empirical Discrete Distribution

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$



Example of Empirical Distribution

- There are four pumps(i-1 to 4) at a petrol station. And on average 1/3 of the customers used pump 1, 1/6 of them used pump 2, another 1/3 used pump 3 and the remaining 1/6 used pump 4. Outline a procedure for selection of pumps by various customers.

i	1	2	3	4
P(i)	1/3	1/6	1/3	1/6
F(i)	1/3	½	5/6	1

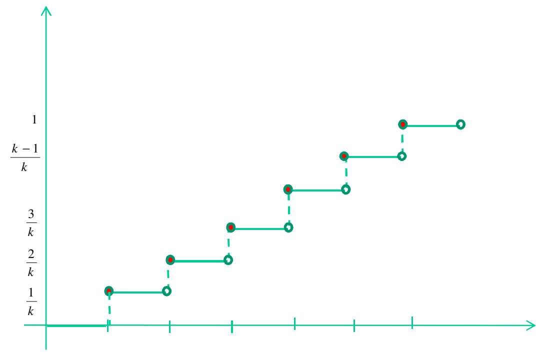
- The procedure:
- Generate r ~U(0,1)

  If $0 \le r < 1/3$ select pump 1
  If $1/3 \le r < 1/2$ select pump 2
  If $1/2 \le r < 5/6$ select pump 3
  If $5/6 \le r < 1$ select pump 4

Discrete Uniform Distribution

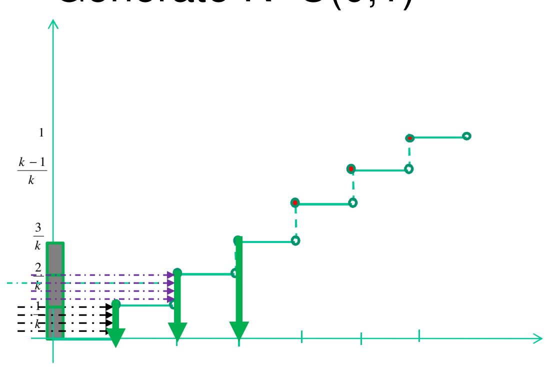
- Consider $p(x) = \frac{1}{k},\ x = 1, 2, 3 \ldots k$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{k}, & 1 \le x < 2 \\ \frac{2}{k}, & 2 \le x < 3 \\ \vdots & \vdots \\ \frac{k-1}{k}, & k-1 \le x < k \\ 1, & k \le x \end{cases}$$



Discrete Uniform Distribution

- Generate R~U(0,1)



$$0 \le R \le \frac{1}{k}$$

$$\frac{1}{k} < R \le \frac{2}{k}$$

$$\frac{2}{k} < R \le \frac{3}{k}$$

$$\frac{3}{k} < R \le \frac{4}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

Discrete Uniform Distribution

$$0 \le R \le \frac{1}{k}$$

$$\frac{1}{k} < R \le \frac{2}{k}$$

$$\frac{2}{k} < R \le \frac{3}{k}$$

$$\frac{3}{k} < R \le \frac{4}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

$$\frac{i-1}{k} < R \le \frac{i}{k}$$

$$\Rightarrow i-1 < Rk \le i$$

$$\Rightarrow i < Rk+1 \text{ and } Rk \le i$$

$$\Rightarrow Rk \le i < Rk+1$$

$$\Rightarrow i = \lfloor RK \rfloor = \text{output } X$$

$$\therefore X = \lfloor RK \rfloor$$

Discrete Uniform Distribution

- Algorithm to generate random variate for p(x)=1/k where x = 1, 2, 3, … k
  - Generate R~U(0,1) uniform random number
  - Return $\lfloor RK \rfloor$

Composition

- Is applied when the distribution F can be expressed as a combination of other distributions F_1, F_2,…. F_n

$$F(x) = \sum_{j=1}^{\infty} p_j F_j(x),$$

$$\sum_{j=1}^{\infty} p_j = 1, p_j \geq 0$$

- Equivalent to say if X has density function f such that $f(x) = \sum_{j=1}^{\infty} p_j f_j(x),$

That corresponds to decomposing f into its convex combination representation

Composition

Geometric Interpretation

- The decomposition can also be seen as dividing the area under ***f*** into regions of areas p_1,p_2,….p_n (since probability distribution corresponds to the area)
- Then determine F_j for each j then apply the inverse method on each one.
- Algorithm
  1. Generate a positive random integer, such that $P(J=j)=p_j$
  2. Return X with distribution F_j

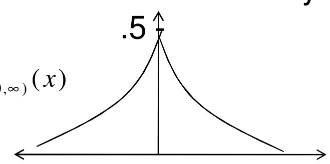
Composition Example1

- For ***0<a<1***, the right trapezoidal distribution has density

$$f(x) = 0.5e^{|x|} \qquad \forall x \quad real$$

Decompose $f(x) = .5e^{x} I_{(-\infty,0)}(x) + .5e^{-x} I_{[0,\infty)}(x)$

$$I_A = \begin{Bmatrix} 1 & x \in A \\ 0 & O.w. \end{Bmatrix}$$



$$f_1(x) = e^{x} I_{[0,1]}(x)$$

$$f_2(x) = e^{-x} I_{[0,1]}(x)$$

$$U_1 = F_1(x) = e^{x}, \quad U_2 = F_2(x) = e^{-x}$$

Use the inverse method

$$x = \ln U_1 \quad or \quad x = \ln U_2$$

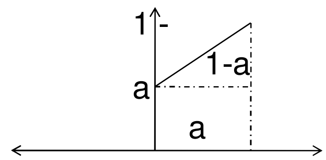
Composition Example1

- Algorithm:
- 1) Generate $U_1 \sim U(0,1), U_2 \sim U(0,1)$
- If U_1 <.5 return $x = \ln U_2$
- U_1 >=.5 $x = \ln U_2$

Composition Example2

- For ***0<a<1***, the right trapezoidal distribution has density

$$f(x) = \begin{cases} a + 2(1-a)x, 0 \le x \le 1 \\ 0, otherwise \end{cases},$$



- We may divide the area as shown
- ***f(x)*** can be decomposed as

$$f(x) = aI_{[0,1]}(x) + (1-a)2xI_{[0,1]}(x)$$

$f_1(x) = I_{[0,1]}(x)$ is U(0,1) density and

$f_2(x) = 2xI_{[0,1]}(x)$ is a right triangular density

$$p_1 = a,\ p_2 = 1-a,\ p_1 + p_2 = 1$$

Composition Example2

- ***Algorithm: Generate*** $U_1 \sim U(0,1), U_2 \sim U(0,1)$
- ***If***

$$U_1 < a \ return \ x = U_1$$

$$U_2 \ge a \quad f_2 = 2x$$

$$U = F_2(x) = x^2$$

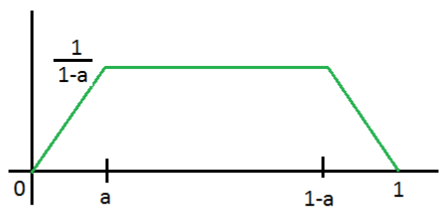
$$x = \sqrt{U_2}$$

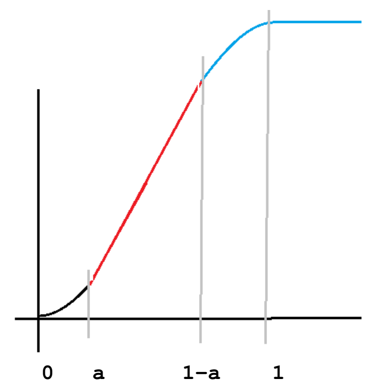
$$return \ \sqrt{U_2}$$

- ***Yet, in some applications, we find computing the square root is expensive So we may use another random number instead U_3 and return x=max(U_2, U_3)***

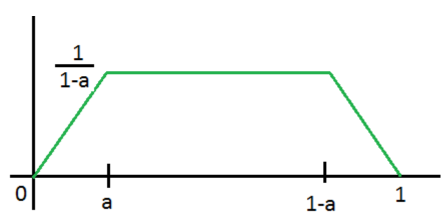
Example 3

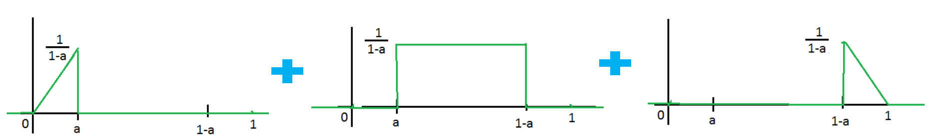
$$f(x)=\begin{cases} 0,\ x\le 0 \\ \dfrac{x}{a(1-a)},0\le x\le a \\ \dfrac{1}{1-a},\ \ a\le x\le 1-a \\ \dfrac{1-x}{a(1-a)},1-a\le x\le 1 \\ 0,\ \ x\ge 1 \end{cases}$$





Example 3





$$f(x)=\frac{x}{a(1-a)}$$

$$F(x)=\frac{x^2}{2a(1-a)}$$

$$area=\frac{a/2}{1-a}$$

$$f(x)=\frac{1}{(1-a)}$$

$$F(x)=\frac{x}{(1-a)}$$

$$area=\frac{1-2a}{1-a}$$

$$f(x)=\frac{1-a}{a(1-a)}$$

$$F(x)=\frac{(1-a)x}{(1-a)}$$

$$area=\frac{a/2}{1-a}$$