Noise Overview

• The phenomenon of noise and its effect on analog circuits.

• Noise characteristics in the frequency and time domains.
  – Thermal noise
  – Shot noise (in BJT)
  – Flicker noise

• Methods of representing noise in circuits.

• Noise in single-stage and differential amplifiers.
Noise is a random process, which means the value of noise cannot be predicted at any time.

How can we incorporate noise in circuit analysis? This is accomplished by observing the noise for a long time and using the measured results to construct a “statistical model” for the noise. While the instantaneous amplitude of noise cannot be predicted, a statistical model provides knowledge about some other important properties of the noise that prove useful and adequate in circuit analysis.
Average Power of Random Signals

Since the signal are not periodic, the measurement must be carried out over a long time:

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \frac{x^2(t)}{R_L} dt$$

where $x(t)$ is a voltage quantity.

Low-power random signal

High-power random signal
Average Noise Power

To simplify calculations, we write the definition of $P_{av}$ as

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t)dt$$

where $P_{av}$ is expressed in $V^2$ rather than W.

In analogy with deterministic signals, we can also define a root-mean-square (rms) voltage for noise as $\sqrt{P_{av}}$. 
Noise Spectrum

- Calculation of noise spectrum

- Power spectral density (PSD):
  The spectrum shows how much power the signal carries at each frequency. More specifically, the PSD, $S_X(f)$, of a noise waveform $x(t)$ is defined as the average power carried by $x(t)$ in a one-hertz bandwidth around $f$. $S_X(f)$ is expresses in $V^2/\text{Hz}$.

- We can apply $x(t)$ to a bandpass filter with center frequency $f_1$ and 1-Hz bandwidth, square the output, and calculate the average over a long time to obtain $S_X(f_1)$. Repeating the procedure for different center frequencies, we arrive at the overall shape of $S_X(f)$.

- It is also common to take the square root of $S_X(f)$, expressing the result in $V/\sqrt{\text{Hz}}$.

- In summary, the spectrum shows the power carried in a small bandwidth at each frequency revealing how fast the waveform is expected to vary in the time domain.
Noise Shaping

- White spectrum (white noise)

- Noise shaping by a transfer function

\[
S_y(f) = S_x(f) |H(f)|^2
\]

- Example: Spectral shaping by telephone BW
Spectrum Power

• Two-sided and one-sided noise spectra

Since $S_X(f)$ is an even function of $f$ for real $x(t)$, the total power carried by $x(t)$ in the frequency range $[f_1, f_2]$ is equal to

$$P_{f_1, f_2} = \int_{-f_2}^{-f_1} S_X(f) df + \int_{f_1}^{+f_2} S_X(f) df = 2\int_{f_1}^{+f_2} S_X(f) df$$

• Folded white spectrum
Amplitude Distribution

- **Probability density function (PDF)**
  - Probability density function (PDF): By observing the noise waveform for a long time, we can construct a “distribution” of the amplitude, indicating how often each value occurs. The distribution of $x(t)$ is defined as
    \[ pdf(x)dx = \text{probability of } x < X < x + dx, \]
  - where $X$ is the measured value of $x(t)$ at some point in time.
  - Gaussian PDF is defined as
    \[ pdf(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right), \]
    where $\sigma$ and $m$ are the standard deviation and mean of the distribution, respectively.
Correlated and Uncorrelated Sources

We add two noise waveforms and take average of the resulting power:

\[
P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left[ x_1(t) + x_2(t) \right]^2 dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x_1^2(t) dt + \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x_2^2(t) dt + \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} 2 x_1(t) x_2(t) dt
\]

\[
= P_{av1} + P_{av2} + \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} 2 x_1(t) x_2(t) dt
\]

**correlation**
The thermal noise of a resistor $R$ can be modeled by a series voltage source, with the one-sided spectral density

$$\overline{V_n^2} = S_v(f) = 4kTR, \quad f \geq 0,$$

where $k = 1.38 \times 10^{-23}$ J/K is the Boltzmann constant and $S_v(f)$ is expressed in $V^2$/Hz.
Resistor Thermal Noise (2/3)

- Example: low-pass filter

We compute the transfer function from $V_R$ to $V_{out}$: 
$$ \frac{V_{out}(s)}{V_R} = \frac{1}{RCs + 1} $$

From the theorem, we have 
$$ S_{out}(f) = S_R(f)\left[\frac{V_{out}}{V_R}(f)\right]^2 = 4kTR\frac{1}{4\pi^2 R^2 C^2 f^2 + 1}. $$

The total noise power at the output:
$$ P_{n, out} = \int_0^\infty \frac{4kTR}{4\pi^2 R^2 C^2 f^2 + 1} df = \frac{2kT}{\pi C} \tan^{-1} \left[ \frac{u = \infty}{u = 0} = \frac{kT}{C} \right] (V^2) $$
Resistor Thermal Noise (3/3)

• Representation of resistor thermal noise by a current source

\[ \overline{I_n^2} = \frac{V_n^2}{R^2} = \frac{4kT}{R} \quad (\text{A}^2/\text{Hz}) \]

• Example

➢ Since the two noise sources are uncorrelated, we add the powers:

\[ \overline{I_{n,\text{tot}}^2} = \overline{I_{n1}^2} + \overline{I_{n2}^2} = 4kT \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

➢ The equivalent noise voltage is given by

\[ \overline{V_{n,\text{tot}}^2} = \overline{I_{n,\text{tot}}^2} \left( R_1 \parallel R_2 \right)^2 = 4kT \left( R_1 \parallel R_2 \right) \]
MOSFET Thermal Noise

• Thermal noise of a MOSFET
  For the MOS devices operating in saturation, the channel noise can be modeled by a current source connected between the drain and source terminals with a spectral density: $\overline{I_n^2} = 4kT\gamma g_m$, where $\gamma$ is equal to 2/3 for long-channel transistors and may be a large value (2) for submicron MOSFETs.

• Output noise voltage
  $$\overline{V_n^2} = \overline{I_n^2}r_o^2 = 4kT\gamma g_m r_o^2$$
Flicker Noise (1/2)

- Dangling bonds at the oxide-silicon interface

- As charge carriers move at the interface, some are randomly trapped and later released by such energy states, introducing *flicker* noise in the drain current.

- The flicker noise is modeled as a voltage source in series with the gate and roughly given by
  \[ \overline{V_n^2} = \frac{K}{C_{ox}WL} \cdot \frac{1}{f} \]
  where \( K \) is a process-dependent constant on the order of \( 10^{-25} \text{ V}^2\text{F} \).

- The flicker noise is also called \( 1/f \) noise, and it does not depend on the bias current or the temperature.

- It is believed that PMOS devices exhibit less \( 1/f \) noise than NMOS transistors because the former carry the holes in a “buried channel”, i.e., at some distance from the oxide-silicon interface.
Flicker Noise (2/2)

• Concept of flicker noise corner frequency

For $1/f$ noise, the drain noise current per unit bandwidth is

$$I_{n,1/f}^2 = V_{n,1/f}^2 \cdot g_m^2 = \frac{K}{C_{ox}WL} \cdot \frac{1}{f} \cdot g_m^2.$$ 

For the thermal noise, the drain noise current per unit bandwidth is

$$I_{n,th}^2 = 4kT \left( \frac{2}{3} g_m \right).$$

Thus, the $1/f$ noise corner frequency, $f_C$, of the output current is determined by

$$f_C = \frac{K}{C_{ox}WL} \cdot g_m,$$

which depends on device dimensions and bias current.
MOS Noise

\[ \overline{I_n^2} = 4kT \gamma g_m \]

\[ \overline{V_n^2} = \frac{4kT \gamma}{g_m} \]

\[ \overline{v_n^2} = \frac{4kT \gamma}{g_m} + \frac{K}{C_{ox} W L} \cdot \frac{1}{f} (V^2 / \text{Hz}) \]
Lemma

Assignment 4a: Prove this Lemma

\[ I_{n,\text{out1}} = \frac{I_n}{Z_S(g_m + 1/r_o) + 1} \]

and that of Fig. 7.33(d) is

\[ I_{n,\text{out2}} = \frac{g_mV_n}{Z_S(g_m + 1/r_o) + 1}. \]

**Lemma** The circuits shown in Fig. 7.33(a) and (b) are equivalent at low frequencies if \( V_n^2 = I_n^2/g_m^2 \) and the circuits are driven by a finite impedance.
Noise in Circuits (1/2)

• How to quantify the effect of noise?

The natural approach would be to set the input to zero and calculate the total noise at the output due to various sources of noise in the circuit.

\[
\overline{V_{n,\text{out}}}^2 = \left( \overline{I_{n,\text{th}}}^2 + \overline{I_{n,1/f}}^2 + \overline{I_{n,R_D}}^2 \right) R_D^2 \\
= \left( 4kT \frac{2}{3} g_m + \frac{K}{C_{\text{ox}} W L} \cdot \frac{1}{f} \cdot g_m^2 + \frac{4kT}{R_D} \right) R_D^2
\]
If the voltage gain is $A_v$, then we have $V_{n,\text{out}}^2 = A_v^2 V_{n,\text{in}}^2$, that is, the input-referred noise voltage is given by the output noise voltage divided by the gain.

The input-referred noise indicates how much the input signal is corrupted by the circuit’s noise, i.e., how small an input the circuit can detect with acceptable SNR.

The input-referred noise is a fictitious quantity in that it cannot be measured at the input of the circuit.
Common-Source Stage (1/3)

Voltage Amplification

Current Generation

How can we reduce the input-referred noise voltage? It implies that the transconductance of \( M_1 \) must be maximized.

The transconductance must be maximized if the transistor is to amplify a voltage signal applied to its gate [Fig.(a)] whereas it must be minimized if the transistor operates as a current source [Fig.(b)].
Common-Source Stage (2/3)

- **Example:** calculate the input-referred thermal noise voltage of the amplifier

Thermal noise:

\[ \overline{V_{n,\text{out}}}^2 = 4kT \left( \frac{2}{3} g_{m1} + \frac{2}{3} g_{m2} \right) (r_{o1}||r_{o2})^2 \]

Voltage gain:

\[ |A_V| = g_{m1}(r_{o1}||r_{o2}) \]

The total noise voltage referred to the gate of \( M_1 \) is

\[ \overline{V_{n,\text{in}}}^2 = 4kT \left( \frac{2}{3} g_{m1} + \frac{2}{3} g_{m2} \right) \frac{1}{g_{m1}^2} = 4kT \left( \frac{2}{3} g_{m1} + \frac{2g_{m2}}{3g_{m1}} \right) \]

It reveals the dependence of \( \overline{V_{n,\text{in}}}^2 \) upon \( g_{m1} \) and \( g_{m2} \), confirming that \( g_{m2} \) must be minimized because \( M_2 \) serves as a current source.
• How to design a CS stage for low-noise operation?

\[
\overline{V_{n,\text{in}}}^2 = \frac{V_{n,\text{out}}^2}{A_v^2} = 4kT\left(\frac{2}{3g_m} + \frac{1}{g_m^2 R_D}\right) + \frac{K}{C_{ox}WL f} \]

- For thermal noise, we must maximize \(g_m\) by increasing the drain current or the device width. A higher \(I_D\) translates to greater power dissipation and limited output voltage swings while a wider device leads to larger input and output capacitance. We can also increase \(R_D\), but at the cost of limiting the voltage headroom and lowering the speed.
- For \(1/f\) noise, the primary approach is to increase the area of the transistor. If \(WL\) is increased while \(W/L\) remains constant, then the device \(g_m\) and its thermal noise do not change but the device capacitances increase.
- These observations point to the trade-offs between noise, power dissipation, voltage headroom, and speed.
Since the input impedance of the source follower is quite high, the input-referred noise current can usually be neglected for moderate driving source impedance.

Compute the input-referred thermal noise:

\[ V_{n,\text{out}}^2 |_{M_2} = I_{n2}^2 \left( \frac{1}{g_{m1}} \right) \left( \frac{1}{g_{mb1}} \right) r_{o1} r_{o2}^2 \]  

(thermal noise)

\[ A_v = \frac{1}{g_{mb1}} r_{o1} r_{o2} \]

\[ V_{n,\text{in}}^2 = V_{n1}^2 + \frac{V_{n,\text{out}}^2 |_{M_2}}{A_v^2} = 4kT \frac{2}{3} \left( \frac{1}{g_{m1}} + \frac{g_{m2}}{g_{m1}} \right) \]
Compute the input-referred $1/f$ noise:

$$\overline{V_{n,\text{out}}^2} = \frac{K}{C_{\text{ox}}(WL)_1} \frac{1}{f} \left(g_{m1}R_{out}\right)^2 + \frac{K}{C_{\text{ox}}(WL)_2} \frac{1}{f} \left(g_{m2}R_{out}\right)^2$$

and

$$R_{out} = \begin{pmatrix} 1 & 1 \\ g_{m1} & g_{mb1} \\ r_{o1} & r_{o2} \end{pmatrix}, \quad |A_v| = g_{m1}R_{out}$$

then

$$\overline{V_{n,\text{in}}^2} = \frac{V_{n,\text{out}}^2}{|A_v|^2} = \frac{K}{C_{\text{ox}}f} \left( \frac{1}{(WL)} + \frac{g_{m2}^2}{(WL)g_{m1}} \right)$$
Input-Refereed Noise

- Representation of input-referred noise by voltage and current sources

- Calculation of input-referred noise voltage (source impedance = 0)

- Calculation of input-referred noise current (source impedance = ∞)
Since the input impedance of the source follower is quite low, the input-referred noise current cannot be neglected especially for high driving source impedance.

\[
\left(4kT \frac{2}{3}g_m + \frac{4kT}{R_D}\right) R_D^2 = \overline{V_{n,in}^2 (g_m + g_{mb})^2} R_D^2.
\]

\[
\overline{V_{n,in}^2} = \frac{4kT(2g_m/3 + 1/R_D)}{(g_m + g_{mb})^2}.
\]
Common Gate Stage (2/2)

- Equate the output noise voltage when the input is short circuit and divide by $A_v$

- Equate the output noise voltage when the input is open circuit and divide by output impedance

$$\overline{I_{n, in}^2} = \frac{4kT}{R_D}$$
M₁, R_D: Since the noise currents of M₁ and R_D flow through R_D, the noise contributed by these two devices is quantified as in a CS stage:

$$V_{n, in}^2_{M₁, R_D} = 4kT\left(\frac{2}{3g_{m1}} + \frac{1}{g_{m1}^2R_D}\right)$$

M₂: In Fig.(b), if the channel length modulation in M₁ is neglected, then $I_{n2} + I_{D2} = 0$, and hence M₂ does not affect $V_{n, out}$.

In Fig.(c), the voltage gain from $V_{n2}$ to the output is quite small if the impedance at node X is large. At high frequencies, the total capacitance at node X, $C_X$, gives rise to a gain:

$$\frac{V_{n, out}}{V_{n2}} \approx \frac{-R_D}{1/g_{m2} + 1/(C_Xs)}$$

increasing the output noise.

At high frequencies, noise of cascode devices start to show up.

- Capacitor at sources of cascode devices shorts this point to ground → Gain from cascode to output increases at high frequencies → Effect of noise increases.
Differential Pairs (1/2)

- Differential pair circuit
- $I_{SS}$ contribute common-mode noise only
- Circuit including input-referred noise source

For low-frequency operation, the magnitude of $\overline{I_{n,\text{in}}^2}$ is typical negligible.
Differential Pairs (2/2)

- Calculation of input-referred noise of a differential pair

With the inputs shorted together, we have

\[ V_{n,\text{out}}|_{M_1} = \frac{I_{n1}}{2} R_{D1} + \frac{I_{n1}}{2} R_{D2} \]

\[ V_{n,\text{out}}^2|_{M_1} = I_{n1}^2 R_D^2 \quad \text{(if} \quad R_{D1} = R_{D2} = R_D) \]

Similarly, \[ V_{n,\text{out}}^2|_{M_2} = I_{n2}^2 R_D^2 \].

\[ \Rightarrow \quad V_{n,\text{out}}^2|_{M_1,M_2} = (I_{n1}^2 + I_{n2}^2) R_D^2 \]

Taking into account the noise of \( R_{D1} \) and \( R_{D2} \), the total output noise:

\[ V_{n,\text{out}}^2|_{M_1,M_2} = (I_{n1}^2 + I_{n2}^2) R_D^2 + 2(4kT R_D) \]

\[ = 8kT \left( \frac{2}{3} g_m R_D^2 + R_D \right) \]

And, \(|A_v| = g_m R_D\), we have

\[ V_{n,\text{in}}^2 = \frac{V_{n,\text{out}}^2}{|A_v|^2} = 8kT \left( \frac{2}{3g_m} + \frac{1}{g_m^2 R_D} \right) \approx 2V_{n,\text{in}}^2|_{\text{CS stage}} \]
Noise Bandwidth

- Output noise spectrum of a circuit

The total output noise:

\[
\overline{V_{n,\text{out},\text{tot}}} = \int_0^\infty \overline{V_{n,\text{out}}} \, df
\]

and

\[
V_0^2 \cdot B_n = \int_0^\infty \overline{V_{n,\text{out}}} \, df
\]

Noise bandwidth \((B_n)\): \(B_n\) allows a fair comparison of circuits that exhibit the same low-frequency noise, \(V_0^2\), but different high-frequency transfer functions.
Assignment 4b:

- Calculate the input referred thermal noise voltage (neglect both channel and body effects)
Assignment 4b (cont):