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STEP-STRESS RELIABILITY FOR GENERALIZED RAYLEIGH DISTRIBUTION BASED ON CENSORING DATA

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Abstract

In modern times, accelerated testing of products has become an integral part of manufacturing industry in order to gauge the life span of the products under simulated stresses. Present article focuses on the step-stress partially accelerated life tests (SS-PALT) when the lifetime of a product follows generalized Rayleigh distribution (GRD).

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Maximum likelihood (ML) estimates and the acceleration factor (AF) are obtained based on type II censoring scheme. Statistical properties of the estimates, with two sets of true parameters, were studied in terms of relative absolute bias (RABias), mean square error (MSE) and relative error (RE). Moreover, variance-covariance (var-cov) matrices of the estimated parameters were produced and confidence intervals for two different levels were also presented in tabulated form. All estimates of the true parameters were obtained using iterative process. Large/small properties hold good for the proposed SS-PALT with GRD.

1. Introduction

In reliability analysis, Rayleigh distribution has played a significant role during the last two decades. Rayleigh in 1880 derived this distribution while solving some problems in acoustics field. An extension of the Rayleigh distribution is its generalized form known as generalized Rayleigh distribution (GRD) and is also named as Burr type X distribution with two parameters (scale and shape). For skewed data sets, GRD is the most effective distribution, reason being, the density function of the GRD is right skewed. Surles and Padgett [28] suggested in their study on the GRD posited that the parameters of GRD can also be used in modeling strength and general life time of the products.

Product's lifetime is the most important quality variable. The lifetime depends on the effects of various factors, e.g., the working of the product under different stress levels, and also under varying temperature, voltage, and load. In accelerated testing, the stress levels on the testing units are increased to study the changes in the parameters of the life distributions which under normal stress would not be observed as quickly as under the accelerated testing. Hence, for obtaining quick know-how regarding the lifetime of a product, accelerated life testing (ALT) is carried out mostly in manufacturing concerns. ALT, in fact, means that the testing units are subjected to a little higher stress than the normal stress or the testing units are studied under the normal stress to observe the lifetime of the units. If the mathematical model of the lifetime of testing units is known or can be assumed in advance, then such ALT is called ordinary ALT (OALT). But, in some real life situations, mathematical models of the testing units are unknown or are difficult to be assumed. Then the researchers resort to partial ALT (PALT). Here, again the testing units are subjected to both accelerated and normal stresses to observe the lifetime of products.

Nelson [21] posited that, stress can be applied in a variety of ways. But the present study is mainly concerned with step-stress, which means increasing the stress with specified discrete sequence on the testing units for testing the product. In situations where the mathematical model related to testing conditions of mean lifetime of the products is unknown, it cannot be assumed that for obtaining rapid information for the lifetime of high reliability products, the experimenters resort to the step-stress PALT (SS-PALT). In applying simple step-stress scheme, units are subjected to a certain amount of stress and the same will be increased at some pre-determined time. But the word ‘simple’ here signifies that only two stress levels are used in a test. Specifically, a unit is put under a predetermined low stress for a predetermined time and if the test unit does not fail, then the stress is increased and again the unit is held under the new stress for a specified time. This process continues until the unit under test fails or censoring scheme is arrived at.

Many authors have worked on the SS-PALT in the last three decades. Parameter estimation of exponential distribution under type I censoring using ML method, along with the estimation of AF were elaboratively covered by Bai and Chung [10]. Development of optimal ALT for Burr type XII distribution with type I censoring under periodic inspection was done by Ahmad and Islam [8]. Weibull distribution which is the most popular distribution used in failure rate situations was used for SS-PALT under type I censoring for estimating the parameters and accelerating factor using ML method by Attia et al. [9]. Furthering the work of Attia et al. [9], a study was undertaken by Abdel-Ghaly et al. [2] to estimate the parameters and AF of Weibull distribution under both type I and type II censored data, using ML method. Same group of authors in 2003 concurrently studied the estimation and optimal design problems under SS-PALT with type I and type II censoring for the compound Pareto distribution, Abdel-Ghaly et al. [3]. Estimation problem of log-logistic distribution parameters under SS-PALT was carried out by Abdel-Ghani [4]. Ismail [15] deliberated on the estimation and optimal design problems for the Gompertz distribution in SS-PALT under type I censoring. An upsurge is witnessed in the subject area in the last decade, for instance, a study of SS-PALT was considered by Abd-Elfattah et al. [1] for the estimation of parameters and AF using ML method for Burr type XII distribution under type I censoring. SS-PALT, when the lifetime of the items under study followed a finite mixture of distributions, was studied by Abdel-Hamid and Al-Hussaini [5]. The same group of authors in 2009 also studied exponentiated exponential distribution for SS-PALT when the censoring was of type I. For more on the topic of SS-PALT, the readers are referred to Gouno et al. [13], Li and Fard [19], Xu and Fei [30], Yang [31], Balakrishnan et al. [11], Nelson [22], Ma and Meeker [20] and Wu et al. [29]. Going through the literature, the authors could not come across any study applying SS-PALT on GRD. Hence, emerges the important idea for carrying out the present work. The current study will focus on the estimation of generalized Rayleigh distribution parameters and AF in step-stress accelerated life testing using maximum likelihood method for type II censored data. The statistical performance of the parameters estimated through ML methods would be assessed in terms of relative absolute bias (RABias), mean square error (MSE) and the behaviour of standard errors (SE). Moreover, the confidence intervals for the estimated parameters will also be developed.

Rest of the paper is formatted as: GR model and the method for testing units will elaboratively be explained in Section 2; Section 3 presents point and interval estimation of parameters and AF for the GRD under type II censoring using ML method. Mathematical equations for obtaining variance-covariance (var-cov) matrix are developed in Section 4. For practical demonstration of the proposed techniques, a simulation study with two sets of parameters is carried out in Section 5. Discussion regarding the results and a brief conclusion are given in Section 6.

2. The Proposed Model

This section introduces the assumed model for product life and also fully describes the test method.

Let X be a random variable with scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$. Then the cumulative distribution function (CDF) of X is given by

$$F(t, \alpha, \lambda) = (1 - e^{-\lambda t^2})^{\alpha}, \quad t > 0, \lambda > 0, \alpha > 0. \tag{2.1}$$

Therefore, the probability density function (pdf) has the form

$$f(t, \alpha, \lambda) = 2\alpha\lambda t e^{-\lambda t^2}(1 - e^{-\lambda t^2})^{\alpha-1}, \quad x > 0, \lambda > 0, \alpha > 0. \tag{2.2}$$

The survival function of the GRD is given by

$$R(t, \alpha, \lambda) = 1 - (1 - e^{-\lambda t^2})^{\alpha}, \quad t > 0, \lambda > 0, \alpha > 0. \tag{2.3}$$

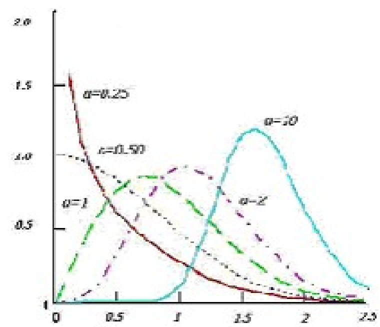

Figure 1. The density functions of the generalized Rayleigh distribution for different values of shape parameter.

Sartawi and Abu-Salih [25], Jaheen [16, 17], Ahmad et al. [7], Raqab and Kundu [23] and Surles and Padgett [27] studied the one-parameter (scale) GRD. Various approaches of estimating the unknown parameters of the GRD were used by Kundu and Raqab [18]. More properties of GRD were explored by the same group of authors in 2006. The shape of the density function for the GRD for different values of the shape parameter $(\alpha = 0.25, 0.5, 1, 2 \text{ and } 10)$ is shown in Figure 1.

Testing units under the step-stress ALT is performed under the normal conditions. In case of non-failure of any units in a predetermined time τ, the testing units are subjected to accelerated conditions till the units fail to work. Sequential switching to the higher level acceleration is continued till the failure of the testing units. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the AF β. In this case, the switching to the higher stress level will shorten the life of test item. Thus, the total lifetime of a test item, denoted by Y, passes through two stages, which are the normal and accelerated conditions. Then the lifetime of the unit in SS-PALT is given as follows:

$$Y = \begin{cases} T & \text{if } T \leq \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{2.4}$$

where T is the lifetime of an item at the condition when used, τ is the stress change time and β is the AF which is the ratio of mean life at normal condition to that at accelerated condition, usually $\beta > 1$. Assuming that the lifetime of the testing units follow GRD with parameters λ and α, the probability density function of total lifetime Y of an item is given by:

$$f(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ f_1(y) & \text{if } 0 < y \leq \tau \\ f_2(y) & \text{if } y > \tau, \end{cases} \tag{2.5}$$

where $f_1(y) = 2\alpha\lambda y e^{-\lambda y^2}(1 - e^{-\lambda y^2})^{\alpha - 1}, \ \ \lambda > 0, \ \ \alpha > 0$ is the equivalent form to equation (2.2) and

$$f_2(y) = 2\alpha\lambda\beta[\tau + \beta(y - \tau)]e^{-\lambda[\tau+\beta(y-\tau)]^2}$$
$$\cdot \{1 - e^{-\lambda[\tau+\beta(y-\tau)]^2}\}^{\alpha-1}, \quad \lambda > 0, \alpha > 0, \beta > 1$$

is obtained by the transformation variable technique using equations (2.2) and (2.4).

3. ML Estimation

In statistics, ML method is one of the most time tested method for estimating the parameters of any distribution. The underlying logic in ML parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. The estimators obtained through the application of ML method are both consistent and asymptotically normally distributed. This section will elaborate the procedure for obtaining the point and interval estimation for the parameters and AF of GRD based on type II censoring using ML method.

Point estimates

In type II censoring, the test terminates when the censoring number of failures r is reached. The observed values of the total lifetime Y are $y_{(1)} \leq \cdots \leq y_{(n_1)} \leq \tau \leq y_{(n_1+1)} \leq \cdots \leq y_{(r)}$, where $r = n_1 + n_2$ is the total number of failures in the test, and n_1 and n_2 are the numbers of items failed at normal and accelerated conditions, respectively. Let δ_{1i}, δ_{2i} be indictor functions such that

$$\delta_{1i} = \begin{cases} 1 & y_{(i)} \leq \tau, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \dots, n$$

and

$$\delta_{2i} = \begin{cases} 1 & \tau < y_{(i)} \leq y_{(r)}, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \dots, n.$$

For simplifying, $y_{(i)}$ can be expressed by y_i. Since the lifetimes $y_1, \dots, y_n$ of n items are independent and identically distributed random variables, their likelihood function is given by

$$L(\underline{y}; \beta, c, k) = \prod_{i=1}^{n} \{f_1(y_i)\}^{\delta_{1i}} \{f_2(y_i)\}^{\delta_{2i}} \{R(y_{(r)})\}^{\bar{\delta}_{1i}\bar{\delta}_{2i}} \tag{3.1}$$

$$L(\underline{y}; \beta, c, k) = \prod_{i=1}^{n} \{2\alpha\lambda y_i e^{-\lambda y_i^2} (1 - e^{-\lambda y_i^2})^{\alpha-1}\}^{\delta_{1i}}$$

$$\times \{2\alpha\lambda\beta Q e^{-\lambda Q^2} \{1 - e^{-\lambda Q^2}\}^{\alpha-1}\}^{\delta_{2i}}$$

$$\times \{e^{-\lambda\alpha E^2}\}^{-\bar{\delta}_{1i}\bar{\delta}_{2i}},$$

where $\bar{\delta}_{1i} = 1 - \delta_{1i}$, $\bar{\delta}_{2i} = 1 - \delta_{2i}$, $Q = [\tau + \beta\{y_i - \tau\}]$,

$$E = [\tau + \beta(y_{(r)} - \tau)], \sum_{i=1}^{n} \delta_{1i} = n_1, \sum_{i=1}^{n} \delta_{2i} = n_2 \text{ and } \sum_{i=1}^{n} \bar{\delta}_{1i}\bar{\delta}_{2i} = n - r.$$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is given by

$$\begin{aligned} \ln L &= r \ln \alpha + r \ln \lambda + n_2 \ln \beta \\ &+ (\alpha - 1)\left\{\sum_{i=1}^{n} \delta_{1i} \ln(1 - e^{-\lambda y_i^2}) + \sum_{i=1}^{n} \delta_{2i} \ln(1 - e^{-\lambda Q^2})\right\} \\ &- \lambda\left\{\sum_{i=1}^{n} \delta_{1i} y_i^2 + \sum_{i=1}^{n} \delta_{2i} Q^2 + \alpha(n - r) E^2\right\} \\ &- \sum_{i=1}^{n} \delta_{1i} \ln(y_i) + \sum_{i=1}^{n} \delta_{2i} \ln Q. \end{aligned} \tag{3.2}$$

ML estimators of β, α and λ are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood to be zero with respect to β, α and λ, respectively. Therefore, the system of equations is as follows:

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} &= \frac{n_2}{\beta} + (\alpha - 1)\sum_{i=1}^{n} 2Q\delta_{2i} e^{\lambda Q^2} (y_i - \tau)(1 - e^{-\lambda Q^2})^{-1} \\ &- 2\alpha(n - r) E(y_{(r)} - \tau) \\ &+ \sum_{i=1}^{n} 2Q\delta_{2i}(y_i - \tau) + \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)(Q)^{-1}, \end{aligned} \tag{3.3}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{n} \delta_{1i} \ln(1 - e^{-\lambda y_i^2}) + \sum_{i=1}^{n} \delta_{2i} \ln(1 - e^{-\lambda Q^2}) \tag{3.4}$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{n} \delta_{1i} y_i^2 - \sum_{i=1}^{n} \delta_{2i} Q^2 - \alpha(n-r)E^2$$

$$+ (\alpha - 1)\left\{\sum_{i=1}^{n} \delta_{1i} y_i^2 e^{\lambda y_i^2} (1 - e^{-\lambda y_i^2})^{-1}\right.$$

$$\left. + \sum_{i=1}^{n} \delta_{2i} Q^2 e^{-\lambda Q^2} (1 - e^{-\lambda Q^2})^{-1}\right\}. \tag{3.5}$$

From equation (3.4), the ML estimate of α is expressed by

$$\hat{\alpha} = \frac{-r}{A}, \tag{3.6}$$

where $A = \sum_{i=1}^{n} \delta_{1i} \ln(1 - e^{-\hat{\lambda} y_i^2}) + \sum_{i=1}^{n} \delta_{2i} \ln(1 - e^{-\hat{\lambda} Q^2} Q^{\hat{c}})$.

Consequently, by substituting $\hat{\alpha}$ into equations (3.3) and (3.5), the system equations are reduced into two nonlinear equations as follows:

$$\frac{n_2}{\hat{\beta}} - \left(\frac{r}{A} + 1\right) \sum_{i=1}^{n} 2Q\delta_{2i} e^{\lambda Q^2} (y_i - \tau)(1 - e^{-\lambda Q^2})^{-1}$$

$$\cdot\, 2\frac{r}{A} \alpha(n-r)E(y_{(r)} - \tau) + \sum_{i=1}^{n} 2Q\delta_{2i}(y_i - \tau)$$

$$+ \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)(Q)^{-1} = 0 \tag{3.7}$$

and

$$\frac{r}{\hat{\lambda}} - \sum_{i=1}^{n} \delta_{1i} y_i^2 - \sum_{i=1}^{n} \delta_{2i} Q^2 + \frac{r}{A}(n-r)E^2$$

$$-\left(\frac{r}{A} + 1\right)\left\{\sum_{i=1}^{n} \delta_{1i} y_i^2 e^{\lambda y_i^2} (1 - e^{-\hat{\lambda} y_i^2})^{-1} + \sum_{i=1}^{n} \delta_{2i} Q^2 e^{-\hat{\lambda} Q^2} (1 - e^{-\hat{\lambda} Q^2})^{-1}\right\} = 0. \tag{3.8}$$

Since the closed form solution to nonlinear equations (3.7) and (3.8) is very hard to obtain, an iterative procedure is applied to solve these equations numerically. Newton-Raphson method is applied for simultaneously solving the nonlinear equations to obtain $\hat{\beta}$ and $\hat{\lambda}$. Therefore, $\hat{\alpha}$ is calculated easily from equation (3.6).

Interval estimates

If $L_\theta = L_\theta(y_1, ..., y_n)$ and $U_\theta = U_\theta(y_1, ..., y_n)$ are functions of the sample data $y_1, ..., y_n$, then a confidence interval for a population parameter θ is given by

$$p[L_\theta \leq \theta \leq U_\theta] = \gamma, \tag{3.9}$$

where L_θ and U_θ are the lower and upper confidence limits which enclose θ with probability γ. The interval $[L_\theta, U_\theta]$ is called a $100\gamma\%$ confidence interval for θ, where $\theta = (\alpha, \lambda, \beta)$ is the parameter space.

For large sample size, the ML estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the approximate $100\gamma\%$ confidence limits for the ML estimate $\hat{\theta}$ of a population parameter θ can be constructed such that

$$p\left[-z \leq \frac{\theta - \hat{\theta}}{\sigma(\hat{\theta})} \leq z\right] = \gamma, \tag{3.10}$$

where z is the $\left[\frac{100(1-\gamma)}{2}\right]$ standard normal percentile. Therefore, the approximate $100\gamma\%$ confidence limits for a population parameter θ can be obtained such that

$$p[\hat{\theta} - z\sigma(\hat{\theta}) \leq \theta \leq \hat{\theta} + z\sigma(\hat{\theta})] = \gamma. \tag{3.11}$$

Then the approximate confidence limits for β, λ and α will be constructed using equation (3.11) with confidence levels 95% and 99%.

4. Asymptotic Var-cov Matrix

The asymptotic variances of ML estimates are given by the elements of the inverse of the Fisher information matrix $I_{ij}(\underline{\theta}) = E\{-\partial^2 \ln L/\partial\theta_i\partial\theta_j\}$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the observed Fisher information matrix is given by $I_{ij}(\underline{\theta}) = \{-\partial^2 \ln L/\partial\theta_i\partial\theta_j\}$, which is obtained by dropping the expectation on operation E (see Cohen [12]). The approximate (observed) asymptotic variance-covariance matrix F for the ML estimates can be written as follows:

$$F = [I_{ij}(\underline{\theta})], \quad i,\ j = 1,\ 2,\ 3 \text{ and } (\underline{\theta}) = (c,\ \beta,\ k). \tag{4.1}$$

The second partial derivatives of the ML function are given as follows:

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial\beta^2} &= \frac{-n_2}{\beta^2} - 2\alpha(n-r)(y_{(r)}-\tau)^2 + \sum_{i=1}^{n} 2\delta_{2i}(y_i-\tau)^2 \\ &\quad + \sum_{i=1}^{n}\delta_{2i}(y_i-\tau)^2(Q)^{-2} + (\alpha-1)\sum_{i=1}^{n}\delta_{2i}(y_i-\tau)^2\{(1-e^{-\lambda Q^2})^{-1} \\ &\quad \cdot [2e^{\lambda Q^2} + 4\lambda Q^2 e^{\lambda Q^2}] - 4\lambda Q^4(1-e^{-\lambda Q^2})^{-2}\},\end{aligned} \tag{4.2}$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = \sum_{i=1}^{n} 2Q\delta_{2i}e^{\lambda Q^2}(y_i-\tau)(1-e^{-\lambda Q^2})^{-1} + 2(n-r)E(y_{(r)}-\tau), \tag{4.3}$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} = (\alpha-1)\sum_{i=1}^{n}\delta_{2i}2Q^3(y_i-\tau)(e^{\lambda Q^2}-2)(1-e^{-\lambda Q^2})^{-2}, \tag{4.4}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} = \sum_{i=1}^{n}\delta_{2i}Q^2e^{\lambda Q^2}(1-e^{-\lambda Q^2})^{-2} + \sum_{i=1}^{n}\delta_{1i}y_i^2e^{-\lambda y_i^2}(1-e^{-\lambda y_i^2})^{-2}, \tag{4.5}$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = \frac{-r}{\alpha^2}, \tag{4.6}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{-r}{\lambda^2} + (\alpha - 1)\sum_{i=1}^{n} \delta_{2i} Q^4 (e^{\lambda Q^2} - 2)(1 - e^{-\lambda Q^2})^{-2}$$

$$+ (\alpha - 1)\sum_{i=1}^{n} \delta_{1i} y_i^4 (e^{\lambda y_i^2} - 2)(1 - e^{-\lambda y_i^2})^{-2}. \tag{4.7}$$

Consequently, the ML estimators of β, λ and α have an asymptotic variance-covariance matrix defined by inverting the Fisher information matrix F and by substituting $\hat{\beta}$ for β, $\hat{\lambda}$ for λ and $\hat{\alpha}$ for α.

5. Simulation Study

Researchers resort to simulation studies to imitate real-life situation and it is resorted to in order to save time and cost where destructive tests are conducted. For illustrating the proposed theoretical results, simulation study was conducted to gauge the performance of the estimated parameters with a predetermined AF. The measures to gauge the performances of the estimated parameters are: (a) relative absolute bias (RABias); which is the absolute difference between the mean estimates and its true value divided by the true value of the parameter; $\left(\text{i.e., } RABias(\hat{\theta}) = \left|\frac{\hat{\theta} - \theta}{\theta}\right|\right)$, (b) mean square error (MSE); which is the sum of squares of the difference between the estimated parameter and its true value divided by the number of the sample (i.e., $MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$), and (c) relative error (RE); which is the square root of mean square error divided by the value of the parameter $\left(\text{i.e., } RE(\hat{\theta}) = \frac{\sqrt{MSE(\hat{\theta})}}{\theta}\right)$. Additionally, the asymptotic variance and covariance (var-cov) matrix and two-sided confidence intervals of the AF and two sets of parameters are obtained. The simulation process is discussed as under:

Step 1. Generating from GRD 1000 random samples of different sizes 20(20)100(100)500, (small, medium and large). Inverse CDF method will be applied to extract random samples, i.e., first equating the CDF of GRD to U where U follows uniform distribution and then working out the expression for generating the samples. The expression for generating random samples using equation (2.1) is $Y = \left[-\frac{1}{\lambda}\ln(1 - U^{\left(\frac{1}{\alpha}\right)})\right]^{0.5}$ which again follows GRD. For the present study, two sets of true parameters will be considered $(\alpha = 0.5, \beta = 1.25, \lambda = 1)$ and $(\alpha = 2, \beta = 1.75, \lambda = 2)$.

Step 2. Under ordinary conditions, selecting the censoring time $\tau = 2$ and let $r = 0.75n$ represent in PALT, the total number of failures.

Step 3. The parameters and the AF under type II censoring were estimated for SS-PALT for the selected sample using the two sets of parameters.

Step 4. Since the likelihood equation for shape parameter α given in equation (3.6) was in closed form, so the estimates were easily worked out. For the numerical solutions of two nonlinear likelihood equations for the parameters β and λ given in equations (3.7) and (3.8), Newton-Raphson method was used.

Step 5. In order to study large sample properties of the estimators and the AF, for all sample sizes mentioned in Step 1, RABias, MSE and RE were calculated using the two sets of the true parameters.

Step 6. Asymptotic var-cov matrix of the estimators with two sets of parameters for different sample sizes was obtained.

Step 7. For the two sets of parameters and the predetermined AF confidence limits were simulated the two confidence levels $\gamma = 0.95$ and 0.99.

Tables 1, 2 and 3 exhibit the simulated results as described from Steps 5 to 7. From the results following observations, viz-e-viz the performance of SS-PALT parameter estimation of GRD can be inferred:

(1) Table 1 shows the values of the three measures RABias, MSE and RE for gauging the statistical properties of estimated parameters with predetermined AF using different sample sizes and with two sets of parameters. It is quite evident that the values for the three measures decrease with the increase in the sample sizes. Also, for the second set of parameters $(\alpha = 2, \beta = 1.75, \lambda = 2)$, the ML estimators have performed better than the first set of parameters $(\alpha = 0.5, \beta = 1.25, \lambda = 1)$ for all sample sizes.

(2) Asymptotic var-cov matrices for two sets of parameters and with different sample sizes are displayed in Table 2. Here, again, it is seen that increase in the sample sizes causes a decrease in the values of var-cov matrices.

(3) Table 3 displays the confidence limits for the estimates using two sets of parameters with different sample sizes coupled with confidence levels $\gamma = 0.95$ and 0.99. It is witnessed that with the increase in the sample sizes the interval of the estimators decreases. Hence, a reverse relation. Moreover, increase in the width of the confidence interval is witnessed with the increase in the confidence levels.

6. Discussion/Conclusion

The main focus of the present study was estimating the model parameters and AF of GRD under type II censoring using ML method. Moreover, the two sets of ML estimates of the model parameters with fixed $\tau = 2$ and $r = 0.75n$ were used for evaluating the performance of SS-PALT and model assumptions. The numerical performance of the estimates was assessed using RABias, MSE and RE. If the proposed technique is considered as efficient, then the values of all these three measures - RABias, MSE and RE, must decrease. From the results shown in Table 1, it was witnessed that as the sample size n increased, the values for RABias, MSE and RE decreased. But the decrease in the second set of parameters was more rapid than the first set of parameters which points to the fact that the second set of parameters exhibits superior statistical properties. Consistency and asymptotic normality of the ML estimates can be judged by the decremental effect in the values of variances and covariances of the estimates with the increase in the sample size *n*. Here, again, the second set of parameters exhibits rapid decrease. Precision of the ML estimates was assessed through the values of the confidence intervals of ML estimates at two different confidence levels at $\gamma = 0.99$ and $\gamma = 0.95$. It is witnessed that intervals for $\gamma = 0.99$ were much wider than the intervals for $\gamma = 0.95$ as well. The interval size decreased with increase in the sample size *n*.

Table 1. The RABias, MSE and RE of the parameters $(\alpha, \beta, \lambda, \tau)$ given $r = 0.75n$ for several sample sizes under type II censoring

n	Parameters $(\alpha, \beta, \lambda, \tau)$	(0.5, 1.25, 1, 2)			(2, 1.75, 2, 2)		
		RABias	MSE	RE	RABias	MSE	RE
	α	0.0770	0.0330	0.1210	0.0320	0.0052	0.1040
20	β	0.0870	0.0470	0.1450	0.0120	0.0160	0.1100
	λ	0.1770	0.0110	0.2060	0.0780	0.0066	0.1350
	α	0.0790	0.0270	0.1170	0.0290	0.0049	0.0910
40	β	0.0870	0.0360	0.1270	0.0037	0.0110	0.0920
	λ	0.1800	0.0100	0.2010	0.0840	0.0055	0.1240
	α	0.0830	0.0250	0.1140	0.0270	0.0042	0.0870
60	β	0.0890	0.0330	0.1210	0.0019	0.0081	0.0780
	λ	0.1870	0.0100	0.2000	0.086	0.0049	0.1150
	α	0.0850	0.0220	0.1100	0.0250	0.0041	0.0770
80	β	0.0900	0.0300	0.1160	0.0032	0.0063	0.690
	λ	0.1840	0.0096	0.1960	0.0890	0.0045	0.1120
	α	0.0870	0.0190	0.0970	0.0210	0.0038	0.0690
100	β	0.0940	0.0300	0.1160	0.0046	0.0051	0.0620
	λ	0.1860	0.0095	0.1950	0.0890	0.0043	0.1090
	α	0.0880	0.0170	0.0950	0.0180	0.0033	0.0650
200	β	0.0960	0.0290	0.1140	0.0008	0.0044	0.0580
	λ	0.1870	0.0095	0.1950	0.0860	0.0039	0.1040
	α	0.0895	0.0150	0.0950	0.0170	0.0029	0.0630
300	β	0.0940	0.0280	0.1100	0.0002	0.0041	0.0560
	λ	0.1870	0.0094	0.1930	0.0880	0.0025	0.1040
	α	0.0910	0.0140	0.0930	0.0130	0.0018	0.0610
400	β	0.0960	0.0270	0.1110	0.0003	0.0033	0.0500
	λ	0.1880	0.0093	0.1930	0.0900	0.0039	0.1040
	α	0.0920	0.0120	0.0930	0.0120	0.0018	0.0610
500	β	0.0950	0.0270	0.1090	0.0001	0.0032	0.0500
	λ	0.1880	0.0093	0.1930	0.0911	0.0038	0.1040

Table 2. Asymptotic var-cov of estimates under type II censoring

n	(0.5, 1.25, 1, 2)			(2, 1.75, 2, 2)		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
	1.2160	___	___	0.7260	___	___
20	–0.6700	0.0820	___	–0.0670	0.0570	___
	–0.1540	0.0160	0.0100	–0.0540	0.0007	0.0087
	0.7370	___	___	0.5550	___	___
40	–0.3730	0.0310	___	–0.0300	0.0061	___
	–0.0440	0.0029	0.0042	–0.0240	–0.0003	0.0049
	0.5880	___	___	0.4160	___	___
60	–0.0850	0.0180	___	–0.0160	0.0041	___
	–0.0210	0.0006	0.0026	–0.0130	–0.0004	0.0034
	0.4470	___	___	0.2580	___	___
80	–0.0460	0.0140	___	–0.0120	0.0032	___
	–0.0150	0.0003	0.0019	–0.0094	–0.0004	0.0027
	0.3407	___	___	0.1660	___	___
100	–0.0350	0.0110	___	–0.0095	0.0026	___
	–0.0120	0.0001	0.0016	–0.0076	–0.0004	0.0022
	0.2960	___	___	0.1371	___	___
200	–0.0280	0.0089	___	–0.0074	0.0022	___
	–0.0094	0.00006	0.0013	–0.0060	–0.0004	0.0019
	0.1770	___	___	0.1120	___	___
300	–0.0230	0.0076	___	–0.0063	0.0019	___
	–0.0075	0.00004	0.0011	–0.0051	–0.0003	0.0016
	0.1590	___	___	0.0970	___	___
400	–0.0200	0.0067		–0.0055	0.0017	___
	–0.0067	0.00003	0.0010	–0.0044	–0.0003	0.0014
	0.1370	___	___	0.0786	___	___
500	–0.0180	0.0059	___	–0.0049	0.0015	___
	–0.0058	0.00004	0.0009	–0.0039	–0.0002	0.0013

Table 3. Confidence limits for the parameters at $\gamma = 0.95$ and 0.99

n	Parameters (α, β, λ, τ)	(0.5, 1.25, 1, 2)			(2, 1.75, 2, 2)		
		Standard deviation	Lower bound	Upper bound	Standard deviation	Lower bound	Upper bound
20	α	0.231	0.852	1.264	0.052	0.426	0.915
			0.866	1.349		0.417	0.936
	β	0.180	1.024	1.731	0.126	0.916	1.411
			0.913	1.842		0.838	1.489
	λ	0.053	0.308	0.516	0.066	0.423	0.683
			0.275	0.548		0.382	0.724
40	α	0.203	0.969	1.343	0.042	0.443	0.921
			0.910	1.403		0.420	0.933
	β	0.145	1.093	1.660	0.106	0.947	1.362
			1.004	1.750		0.881	1.428
	λ	0.044	0.324	0.496	0.054	0.443	0.656
			0.297	0.523		0.409	0.690
60	α	0.175	0.988	1.311	0.038	0.471	0.933
			0.936	1.362		0.463	0.954
	β	0.123	1.125	1.609	0.090	0.976	1.329
			1.048	1.685		0.920	1.385
	λ	0.036	0.337	0.476	0.046	0.458	0.639
			0.314	0.499		0.429	0.667
80	α	0.152	1.007	1.289	0.037	0.590	0.775
			0.962	1.334		0.561	0.805
	β	0.110	1.149	1.579	0.079	0.998	1.309
			1.081	1.648		0.949	1.358
	λ	0.033	0.344	0.472	0.041	0.466	0.628
			0.323	0.492		0.440	0.653
100	α	0.131	1.019	1.274	0.034	0.598	0.772
			0.978	1.314		0.570	0.799
	β	0.101	1.161	1.557	0.071	1.016	1.094
			1.098	1.620		0.972	1.338
	λ	0.029	0.350	0.463	0.038	0.473	0.620
			0.332	0.481		0.449	0.644
200	α	0.093	1.023	1.264	0.032	0.600	0.761
			0.984	1.302		0.574	0.786
	β	0.092	1.175	1.536	0.066	1.021	1.281
			1.118	1.593		0.980	1.322
	λ	0.028	0.352	0.461	0.035	0.479	0.618
			0.335	0.478		0.457	0.640
300	α	0.072	1.032	1.253	0.029	0.603	0.759
			0.997	1.288		0.578	0.783
	β	0.087	1.189	1.529	0.064	1.025	1.275
			1.135	1.583		0.985	1.315
	λ	0.025	0.358	0.455	0.032	0.484	0.610
			0.343	0.471		0.464	0.630

400	α	0.059	1.037	1.249	0.027	0.607	0.749
			1.003	1.283		0.585	0.772
	β	0.082	1.195	1.515	0.057	1.035	1.259
			1.144	1.566		1.000	1.294
	λ	0.023	0.361	0.451	0.032	0.484	0.608
			0.347	0.465		0.464	0.628
500	α	0.047	1.042	1.243	0.027	0.606	0.753
			1.011	1.275		0.583	0.776
	β	0.078	1.205	1.509	0.058	1.036	1.264
			1.157	1.557		1.001	1.300
	λ	0.022	0.363	0.450	0.030	0.486	0.605
			0.349	0.463		0.467	0.624

The top value for 95% confidence level and the second value for 99%

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